

# Holographic gauge theories in background fields and surface operators

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**ABSTRACT:** We construct a new class of supersymmetric surface operators in  $\mathcal{N} = 4$  SYM and find the corresponding dual supergravity solutions. We show that the insertion of the surface operator — which is given by a WZW model supported on the surface — appears by integrating out the localized degrees of freedom along the surface which arise microscopically from a D3/D7 brane intersection. Consistency requires constructing  $\mathcal{N} = 4$  SYM in the D7 supergravity background and not in flat space. This enlarges the class of holographic gauge theories dual to string theory backgrounds to gauge theories in non-trivial supergravity backgrounds. The dual Type IIB supergravity solutions we find reveal — among other features — that the holographic dual gauge theory does indeed live in the D7-brane background.

**KEYWORDS:** Gauge-gravity correspondence, Supersymmetric Effective Theories, AdS-CFT Correspondence.

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## 1. Introduction and conclusion

The phase structure of a gauge theory can be probed by studying the behaviour of the order parameters of the theory as we change external parameters, such as the temperature. In order to characterize the possible phases, one may insert an infinitely heavy probe charged particle, and study its response, as it will depend on the phase the gauge theory is in. Known examples of operators inserting such probes are Wilson, Polyakov and 't Hooft operators, which distinguish between the confined, deconfined and the Higgs phase.

It is a natural question to ask whether one can construct an operator which inserts a probe string instead of a probe particle. If so, we can then study the response of the string and analyze whether new phases of gauge theory can be found that are not discriminated by particle probes. Candidate probe strings range from cosmic strings to the wrapped  $D$ -branes of string theory.

Geometrically, an operator inserting a probe string is characterized by a surface  $\Sigma$  in space-time, which corresponds to the worldsheet spanned by the string. One may refer to

such operators as surface operators and will label them by  $\mathcal{O}_\Sigma$ . Such operators are nonlocal in nature and the challenge is to construct them and to understand their physical meaning. For early studies of these operators see [11, 12].

Recently, a class of supersymmetric surface operators in  $\mathcal{N} = 4$  SYM have been constructed by Gukov and Witten [14],<sup>1</sup> while the corresponding gravitational description in terms of smooth solutions of Type IIB supergravity which are asymptotically  $AdS_5 \times S^5$  has been identified in [15]. These operators are defined by a path integral with a codimension two singularity near  $\Sigma$  for the  $\mathcal{N} = 4$  SYM fields. Therefore, these operators are of disorder type as they do not admit a description in terms of an operator insertion which can be written in terms of the classical fields appearing in the Lagrangian.

In this paper we construct a family of surface operators in four dimensional  $\mathcal{N} = 4$  SYM that do admit a description in terms of an operator insertion made out of the  $\mathcal{N} = 4$  SYM fields. In the standard nomenclature, they are order operators. The surface operator is obtained by inserting into the  $\mathcal{N} = 4$  SYM path integral the WZW action supported on the surface  $\Sigma$

$$\exp [iM\Gamma_{\text{WZW}}(A)], \tag{1.1}$$

where:<sup>2</sup>

$$\begin{aligned} \Gamma_{\text{WZW}}(A) = & -\frac{1}{8\pi} \int_{\Sigma} dx^+ dx^- \text{Tr} [(U^{-1}\partial_+U) (U^{-1}\partial_-U) - (U^{-1}\partial_+U) (V^{-1}\partial_-V)] \\ & -\frac{1}{24\pi} \int d^3x \epsilon^{ijk} \text{Tr} [(U^{-1}\partial_iU) (U^{-1}\partial_jU) (U^{-1}\partial_kU)]. \end{aligned}$$

The  $U(N)$  group elements  $U$  and  $V$  are nonlocally related to the  $\mathcal{N} = 4$  SYM gauge field  $A_\mu$  along  $\Sigma$  by:

$$A_+ = U^{-1}\partial_+U \qquad A_- = V^{-1}\partial_-V. \tag{1.2}$$

$M$  is an arbitrary positive integer which labels the level of the WZW model.<sup>3</sup>

We construct these operators by considering the field theory limit of a supersymmetric  $D3/D7$  brane intersection along a two dimensional surface  $\Sigma$ . We find that a consistent description of the low energy dynamics of this brane intersection requires that the gauge theory on the  $D3$ -branes is written down not in flat space *but* in the non-trivial supergravity background created by the  $D7$ -branes.

In this paper we construct this supersymmetric field theory in the  $D7$ -brane supergravity background and show that if we integrate out the degrees of freedom introduced by the  $D7$ -branes that the net effect is to insert the operator (1.2) into the gauge theory action. The same strategy of integrating out the new degrees of freedom introduced on a brane intersection was used in [16] to construct the Wilson loop operators in  $\mathcal{N} = 4$  SYM

<sup>1</sup>These operators play an important role in enriching the gauge theory approach [13] to the geometric Langlands program to the case with ramification.

<sup>2</sup>We note that  $\Gamma_{\text{WZW}}(A)$  differs from the conventional WZW model action by the addition of a local counterterm which is needed to guarantee that the operator has all the appropriate symmetries.

<sup>3</sup>In the string construction of this operator  $N$  denotes the number of  $D3$ -branes while  $M$  is the number of  $D7$ -branes.

and to find the bulk AdS description of a Wilson loop in an arbitrary representation of the gauge group.

The physics responsible for having to consider the gauge theory on the non-trivial supergravity background is that there are chiral fermions localized on  $\Sigma$  arising from the open strings stretching between the  $D3$  and  $D7$  branes. It is well known that the gauge anomalies introduced by these chiral degrees of freedom are cancelled only after the appropriate Chern-Simons terms on the  $D$ -brane worldvolume are included [17]. The Chern-Simons terms needed to cancel the anomalies become non-trivial due to the presence of the RR one-form flux produced by the  $D7$ -branes. We show, however, that it is inconsistent to consider only the RR background produced by the  $D7$ -branes. One must also take into account the non-trivial background geometry and dilaton produced by the  $D7$ -branes as they are of the same order as the effect produced by the RR flux. This can be seen by showing that the gauge theory in flat space in the presence of the Chern-Simon terms does not capture the supersymmetries of the brane intersection. Therefore, we are led to consider the low energy action of  $N$   $D3$ -branes in the supergravity background produced by the intersecting  $D7$ -branes. The gauge theory describing the low energy dynamics preserves eight supersymmetries and is  $ISO(1, 1) \times SU(4)$  invariant.

Given the construction of the surface operator in term of  $D$ -branes we proceed to study the bulk Type IIB supergravity description of these surface operators. We start by showing that there is a regime in the bulk description where the  $D7$ -branes can be treated as probe branes in  $AdS_5 \times S^5$ . We show that this corresponds to the regime where the gauge anomaly is suppressed, the Chern-Simons term can be ignored and the gauge theory lives in flat space. This corresponds to considering the limit where  $g^2 M \ll 1$ , where  $g$  is the gauge theory coupling constant. In this limit the symmetries of the gauge theory are enhanced to the  $SU(1, 1|4)$  supergroup.

We go beyond the probe approximation and construct the exact Type IIB supergravity solutions that are dual to the surface operators we have constructed.<sup>4</sup> These solutions can be found by taking the near horizon limit of the supergravity solution describing the localized  $D3/D7$  brane intersection from which the surface operator is constructed. The dual supergravity solutions take the form of a warped  $AdS_3 \times S^5 \times \mathcal{M}$  metric, where  $\mathcal{M}$  is a two dimensional complex manifold. These solutions also shed light on the geometry where the holographic field theory lives. One can infer that the gauge theory lives on the curved background produced by the  $D7$ -branes by analyzing the dual supergravity geometry near the conformal boundary, thus showing that holography requires putting the gauge theory in a curved space-time. The explicit construction of the supergravity solutions also gives us information about the quantum properties of our surface operators. To leading order in the  $g^2 M$  expansion, the surface operator preserves an  $SO(2, 2) \subset SU(1, 1|4)$  symmetry, which is associated with conformal transformations on the surface  $\Sigma = \mathbb{R}^{1,1}$ . In the probe brane description — where  $g^2 M$  effects are suppressed — we also have the  $SO(2, 2)$  symmetry, while the explicit supergravity solution shows that the  $SO(2, 2)$  symmetry is broken by  $g^2 M$

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<sup>4</sup>The supergravity solution dual to other (defect) operators in  $\mathcal{N} = 4$  have appeared in [18–22, 15, 23–25].

corrections. This shows that  $g^2M$  corrections in the field theory break conformal invariance, which can be seen explicitly by analyzing the gauge theory on the  $D7$ -brane background. This field theory statement is reminiscent<sup>5</sup> to the breaking of conformal invariance by  $g^2M$  effects that occurs when considering  $\mathcal{N} = 4$  SYM coupled to  $M$  hypermultiplets, whose  $\beta$ -function is proportional to  $g^2M$ .

A general lesson that emerges from this work is that gauge theories in non-trivial supergravity backgrounds can also serve as the holographic description of string theory backgrounds. It would be interesting to explore in more detail the dictionary relating bulk and gauge theory computations. An important problem for the future is to understand the physics encoded in the expectation value of surface operators and to determine whether they can be useful probes of new phases of gauge theory. For the surface operators in this paper it would be interesting to compute their expectation value in perturbation theory. Given that these operators are supersymmetric it is conceivable that the computation of their expectation value can be performed in a reduced model, just like the expectation value of supersymmetric circular Wilson loops can be computed by a matrix integral [31, 32]. One may be able to derive the reduced model by topologically twisting the gauge theory by the supercharges preserved by the surface operator. It would also be interesting to compute the expectation value of the surface operator by calculating the on-shell action of the corresponding supergravity solutions.

The plan of the paper is as follows. In section 2 we introduce the  $D3/D7$  brane intersection, the corresponding low energy spectrum and discuss the cancellation of the gauge anomalies via anomaly inflow. We show that the gauge theory on the  $D3$ -branes has to be placed in the supergravity background produced by the  $D7$ -branes and construct explicitly the relevant gauge theory action, derive the appropriate supersymmetry transformations and show that the action has all the required symmetries. We integrate out all the degrees of freedom introduced by the  $D7$ -branes and show that the net effect is to insert the WZW action (1.1) into the  $\mathcal{N} = 4$  SYM path integral. In section 3 we give the bulk description of the surface operators. We show that there is a regime where the  $D7$ -branes can be treated as probe branes in  $AdS_5 \times S^5$  and identify this with the regime in the field theory where the anomaly is suppressed, the Chern-Simons term can be ignored and the gauge theory lives in flat space. We find the explicit exact supergravity solution describing the supergravity background produced by the localized  $D3/D7$  brane intersection and show that in the near horizon limit it is described by an  $AdS_3 \times S^5$  warped metric over a two dimensional manifold. We show that the metric on the boundary, where the gauge theory lives, is precisely the  $D7$ -brane metric on which we constructed the field theory in section 2. Some of technical details and computations are relegated to the appendices.

**Note added:** while this work was being completed, the paper [33] appeared, which also discusses the gauge theory action on the  $D3/D7$  brane intersection and overlaps with section 2.

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<sup>5</sup>Such models have been realized in string theory using brane intersections in e.g. [26, 27]. For attempts at computing the supergravity description of this system see e.g. [28–30].

## 2. Gauge theory and surface operators

### 2.1 Brane intersection and anomalies

The surface operators in this paper are constructed from the low energy field theory on a  $D3/D7$  brane configuration that intersects along a surface  $\Sigma = \mathbb{R}^{1,1}$ . More precisely, we consider the effective description on  $N$   $D3$ -branes with worldvolume coordinates  $x^\mu = (x^0, x^1, x^2, x^3)$  and  $M$   $D7$ -branes whose worldvolume is parameterized by  $(x^0, x^1)$  and  $x^I = (x^4, x^5, x^6, x^7, x^8, x^9)$ . The coordinates that parametrize the surface  $\Sigma$  are  $x^0$  and  $x^1$ :

$$\begin{array}{cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 N D3 & X & X & X & X & & & & & & \\
 M D7 & X & X & & & X & X & X & X & X & X
 \end{array} \tag{2.1}$$

The supersymmetries preserved by the  $D3$ -branes are the following<sup>6</sup>

$$i\gamma^{0123}\epsilon = \epsilon, \tag{2.2}$$

where  $\epsilon$  is a ten dimensional complex Weyl spinor satisfying  $\gamma^{01\dots 89}\epsilon = \epsilon$ , which labels the thirty-two supersymmetries of Type IIB supergravity. The supersymmetries preserved by the  $D7$ -branes are given by:

$$i\gamma^{01456789}\epsilon = \epsilon. \tag{2.3}$$

Therefore, in total there are eight supersymmetries preserved by the brane intersection, which can be shown to be chiral in the two dimensional intersection. If we introduce coordinates

$$x^\pm = x^0 \pm x^1 \quad z = x^2 + ix^3, \tag{2.4}$$

then the unbroken supersymmetries satisfy

$$\gamma_+\epsilon = 0, \tag{2.5}$$

or can alternatively be written as

$$\gamma_{\bar{z}}\epsilon = 0, \tag{2.6}$$

where:

$$\gamma_+ = \frac{1}{2}(\gamma_0 + \gamma_1), \quad \gamma_{\bar{z}} = \frac{1}{2}(\gamma_2 + i\gamma_3). \tag{2.7}$$

In constructing the supersymmetry transformations of the gauge theory living on the brane intersection we will use four dimensional Weyl spinors. In the four dimensional notation, the sixteen supersymmetries preserved by the  $D3$ -branes (2.2) are generated by  $(\epsilon_\alpha^i, \bar{\epsilon}_{\dot{\alpha}i})$ , where  $\epsilon_\alpha^i$  is a four dimensional Weyl spinor of positive chirality transforming in the  $(\mathbf{2}, \mathbf{4})$  representation of  $SL(2, \mathbb{C}) \times SU(4)$  and  $\bar{\epsilon}_{\dot{\alpha}i} = (\epsilon_\alpha^i)^*$ . These spinors generate

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<sup>6</sup>In this paper we denote the  $\gamma$ -matrices in flat space by  $\gamma$ . The curved space  $\gamma$ -matrices are denoted by  $\Gamma$ . They satisfy  $\{\Gamma^M, \Gamma^N\} = 2g^{MN}$ , where  $g_{MN}$  is the space-time metric.

the usual Poincare supersymmetry transformations of  $\mathcal{N} = 4$  Yang-Mills theory. In this notation, the projectors (2.5) and (2.6) can be written as:<sup>7</sup>

$$\tilde{\sigma}_+^{\dot{\alpha}\alpha} \epsilon_\alpha^i = 0, \quad \tilde{\sigma}_z^{\dot{\alpha}\alpha} \epsilon_\alpha^i = 0. \quad (2.8)$$

Therefore, the projections (2.8) imply that  $\epsilon_1^i = \epsilon_2^i$ , which parametrize the eight real supersymmetries preserved by the brane intersection.

In the low energy limit — where  $\alpha' \rightarrow 0$  — massive open strings and closed string excitations decouple and only the massless open strings are relevant. The 3-3 strings yield the spectrum of four dimensional  $\mathcal{N} = 4$  SYM while the quantization of the 3-7 open strings results in two dimensional chiral fermions  $\chi$  localized on the intersection, and transform in the  $(N, \bar{M})$  representation of  $U(N) \times U(M)$ . The massless 7-7 strings give rise to a SYM multiplet in eight dimensions, but these degrees of freedom are non-dynamical in the decoupling limit and appear in the effective action only as Lagrange multipliers.

The action for the localized chiral fermions is given by

$$S_{\text{defect}} = \int dx^+ dx^- \bar{\chi} (\partial_+ + A_+ + \tilde{A}_+) \chi, \quad (2.9)$$

where  $A$  and  $\tilde{A}$  denote the  $D3$  and  $D7$ -brane gauge fields respectively and we have used the coordinates introduced in (2.4). Of the usual Poincare supersymmetries of  $\mathcal{N} = 4$  SYM, whose relevant transformations are given by

$$\delta A_\mu = -i \bar{\lambda}_{\dot{\alpha}i} \tilde{\sigma}_\mu^{\dot{\alpha}\alpha} \epsilon_\alpha^i + \text{c.c.}, \quad \delta \chi = 0, \quad \delta \tilde{A}_\mu = 0, \quad (2.10)$$

the defect term (2.9) is invariant under those supersymmetries for which  $\delta A_+ = 0$ , which are precisely the ones that satisfy the projections in (2.8) arising from the  $D3/D7$  brane intersection.

Quantum mechanically, the path integral over the localized chiral fermions  $\chi$  is not well defined due to the presence of gauge anomalies in the intersection. In order to see how to cure this problem, it is convenient to split the  $U(N)$  and  $U(M)$  gauge fields into  $SU(N) \times U(1)$  and  $SU(M) \times U(1)$  gauge fields. With some abuse of notation, we denote the  $SU(N)$  and  $SU(M)$  parts of the gauge field by  $A$  and  $\tilde{A}$  respectively, while the corresponding  $U(1)$  parts of the gauge field are denoted by  $a$  and  $\tilde{a}$ . Then, the variation of the quantum effective action under an  $SU(N) \times SU(M)$  gauge transformation

$$\delta A_\mu = \partial_\mu L + [A_\mu, L], \quad \delta \tilde{A}_\mu = \partial_\mu \tilde{L} + [\tilde{A}_\mu, \tilde{L}] \quad (2.11)$$

is given by

$$\delta_{L, \tilde{L}} S = \frac{1}{8\pi} \int dx^+ dx^- \left[ M \text{Tr}_{SU(N)} (L dA) + N \text{Tr}_{SU(M)} (\tilde{L} d\tilde{A}) \right], \quad (2.12)$$

so that the theory is anomalous under  $SU(N) \times SU(M)$  gauge transformations. Likewise,  $U(1) \times U(1)$  gauge transformations

$$\delta A_\mu = \partial_\mu l, \quad \delta \tilde{A}_\mu = \partial_\mu \tilde{l}, \quad (2.13)$$

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<sup>7</sup>Our conventions on  $\sigma$ -matrices are summarized in appendix A. They are essentially the same as those in the book [34].

on the quantum effective action yield

$$\delta_{l,\tilde{l}}S = \frac{1}{8\pi} \int dx^+ dx^- NM(l - \tilde{l})(f_{+-} - \tilde{f}_{+-}), \quad (2.14)$$

so that the theory is anomalous under the U(1) gauge transformations generated by  $l - \tilde{l}$ , and where:

$$f = da, \quad \tilde{f} = d\tilde{a}. \quad (2.15)$$

Anomalies supported on  $D$ -brane intersections are cancelled by the anomaly inflow mechanism [17], which relies on the presence of Chern-Simons couplings in the  $D$ -brane worldvolume. The Chern-Simons terms that couple to the SU( $N$ ) and SU( $M$ ) gauge fields are given by

$$S_{\text{CS}}(A) = -\frac{(2\pi\alpha')^2\tau_3}{2} \int G_1 \wedge \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (2.16)$$

and

$$S_{\text{CS}}(\tilde{A}) = -\frac{(2\pi\alpha')^2\tau_7}{2} \int G_5 \wedge \text{Tr} \left( \tilde{A} \wedge d\tilde{A} + \frac{2}{3} \tilde{A} \wedge \tilde{A} \wedge \tilde{A} \right), \quad (2.17)$$

where  $g_s$  is the string coupling constant and  $\tau_3$  and  $\tau_7$  is the  $D3$  and  $D7$ -brane tension respectively:

$$\tau_3 = \frac{1}{g_s(2\pi)^3\alpha'^2}, \quad \tau_7 = \frac{1}{g_s(2\pi)^7\alpha'^4}. \quad (2.18)$$

$G_1$  is the RR one-form flux produced by the stack of  $D7$ -branes and  $G_5$  is the self-dual RR five-form flux produced by the stack of  $D3$ -branes.

In the presence of localized  $D$ -brane sources, the Bianchi identities for the RR fields are modified in a way that the Chern-Simons terms become non-trivial. In our case, the modified Bianchi identities are given by

$$dG_1 = MG_{10}\tau_7\delta^2(z\bar{z}) = g_sM\delta^2(z\bar{z}) \quad (2.19)$$

and

$$dG_5 = NG_{10}\tau_3\delta(x^4)\delta(x^5)\dots\delta(x^9), \quad (2.20)$$

where  $G_{10}$  is the ten-dimensional Newton's constant which is given by:

$$G_{10} = g_s^2(2\pi)^7\alpha'^4. \quad (2.21)$$

Therefore, under an SU( $N$ )  $\times$  SU( $M$ ) gauge transformation (2.11), the Chern-Simons terms (2.16) and (2.17) are not invariant, and reproduce the two-dimensional anomaly

$$\delta S_{\text{CS}}(A) + \delta S_{\text{CS}}(\tilde{A}) = -\frac{1}{8\pi} \int dx^+ dx^- \left[ M\text{Tr}_{\text{SU}(N)}(LdA) + N\text{Tr}_{\text{SU}(M)}(\tilde{L}d\tilde{A}) \right], \quad (2.22)$$

where  $L$  and  $\tilde{L}$  are taken to vanish at infinity. This mechanism provides a cancellation of the SU( $N$ ) and SU( $M$ ) gauge anomalies [17].

The Chern-Simons terms containing the U(1) gauge fields  $a$  and  $\tilde{a}$  are more involved. They have been studied in [35], where the anomalies of a closely related  $D5/D5$  brane



intersection along a two dimensional defect were studied.<sup>8</sup> The analogous terms for the  $D3/D7$  system are given by:

$$S_{\text{CS}}(a, \tilde{a}) = -\frac{(2\pi\alpha')^2\tau_3}{2}N \int G_1 \wedge a \wedge f - \frac{(2\pi\alpha')^2\tau_7}{2}M \int G_5 \wedge \tilde{a} \wedge \tilde{f} \quad (2.23)$$

$$+\frac{(2\pi\alpha')^2\tau_3}{2}N \int G_1 \wedge a \wedge \tilde{f} + \frac{(2\pi\alpha')^2\tau_7}{2}M \int G_5 \wedge \tilde{a} \wedge f.$$

The first two terms are the usual Chern-Simons couplings analogous to (2.16) and (2.17). The third term arises from the familiar coupling on the  $D3$ -brane worldvolume of the form

$$\int a \wedge F_3, \quad (2.24)$$

where  $F_3$  is the RR three-form flux, which as argued in [35] is given by  $F_3 = G_1 \wedge \tilde{f}$  in the presence of  $G_1$  and  $\tilde{f}$  background fields. Note that  $\tilde{f}$  in the third term is to be evaluated at  $x^I = 0$ . Similarly, the last term arises from the Chern-Simons coupling on the  $D7$ -brane

$$\int \tilde{a} \wedge F_7, \quad (2.25)$$

where the RR seven-form flux is now given by  $G_5 \wedge f$ , where  $f$  is to be evaluated at  $z = 0$ . If we now perform a  $U(1) \times U(1)$  gauge transformation, the variation of (2.23) is given by

$$\delta S_{\text{CS}}(a, \tilde{a}) = -\frac{1}{8\pi} \int dx^+ dx^- NM(l - \tilde{l})(f_{+-} - \tilde{f}_{+-}), \quad (2.26)$$

where we have used the modified Bianchi identities (2.19) and (2.20). Therefore, by including all the Chern-Simons couplings all anomalies cancel.

**Field theory construction of gauge theories with anomaly inflow.** Turning on the RR fluxes (2.19) and (2.20) produced by the  $D3$  and  $D7$  branes is crucial in obtaining an effective theory which is anomaly free. Usually, in analyzing the low energy gauge theory on a  $D$ -brane intersection in flat space we can ignore the RR flux produced by the branes. However, whenever there are localized gauge anomalies the RR flux cannot be neglected as it generates the required Chern-Simons needed to cancel the anomaly. But  $D$ -branes also source other supergravity fields, such as the metric and the dilaton. It is therefore inconsistent to study the low energy gauge theory in flat space with only the addition of the RR-induced Chern-Simons terms. Physically, one must consider the gauge theory in

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<sup>8</sup>The physics of that system is quite different from the  $D3/D7$  system studied in this paper. In [35] it was argued that the dynamics of the gauge fields pushes the fermions away from the intersection by a distance determined by the (dimensionful) gauge theory coupling constant. In our system the fermions are stuck at the intersection since the  $U(N)$  coupling constant is dimensionless unlike the one on the  $D5$ -branes which is dimensionful while the  $U(M)$  gauge coupling constant vanishes in the decoupling limit, pinning down the fermions at the intersection. Moreover, in [35] the symmetry is enhanced from  $ISO(1, 1)$  to  $ISO(1, 2)$  while in our system the symmetry is enhanced from  $ISO(1, 1)$  to  $SO(2, 2)$ , but only to leading order in the  $g^2M$  expansion. Here we also resolve a puzzle left over in their paper, which is to construct the gauge theory action with all the expected supersymmetries.

the *full* supergravity background produced by the other  $D$ -brane, as the effect of the metric and dilaton is of the same order as the effect of the RR flux.

One way to see that it is inconsistent to consider the gauge theory on the  $D3$ -branes in flat space and in the presence of only the RR-flux produced by the  $D7$ -branes is to note that the naive action of the system

$$S = S_{\mathcal{N}=4} + S_{\text{defect}} + S_{\text{CS}}(A) + S_{\text{CS}}(\tilde{A}) + S_{\text{CS}}(a, \tilde{a}), \quad (2.27)$$

is not supersymmetric, where  $S_{\mathcal{N}=4}$  is the usual flat space action of  $\mathcal{N} = 4$  SYM and the other terms appear in (2.9), (2.16), (2.17) and (2.23) respectively. In particular, this low-energy gauge theory does not capture the supersymmetries of the brane intersection (2.8), and therefore is not a faithful description of the low energy dynamics.

In the rest of this section we construct the low energy gauge theory living on the  $D3$ -branes when embedded in the full supergravity background of the  $D7$ -branes — which includes the appropriate Chern-Simons terms — and show that the field theory has all the required symmetries.

## 2.2 The $D7$ -brane background

As just argued, we must construct the low energy gauge theory on the  $D3$ -branes when placed in the full supergravity background of the  $D7$ -branes. We will devote this subsection to reviewing the salient features of the  $D7$ -brane background.

The metric produced by the  $D7$ -branes in the brane array (2.1) is given by

$$ds^2 = g_{MN} dx^M dx^N = H_7^{-1/2} (-(dx^0)^2 + (dx^1)^2 + dx^I dx^I) + H_7^{1/2} dz d\bar{z}, \quad (2.28)$$

where the coordinates are defined in (2.1). The RR axion  $C$  and the dilaton  $\Phi$  can be combined into a complex field  $\tau$  with is holomorphic in  $z$ , so that the axion and the dilaton produced by the  $D7$ -branes is given by:

$$\partial_{\bar{z}}\tau = 0 \quad \text{where} \quad \tau = C + ie^{-\Phi} \quad e^{-\Phi} = H_7. \quad (2.29)$$

This background solves the Killing spinor equations of Type IIB supergravity

$$\delta\Psi_M = \partial_M\epsilon + \frac{1}{4}\omega_M^{AB}\Gamma_{AB}\epsilon - \frac{i}{8}e^\Phi\partial_N C\Gamma^N\Gamma_M\epsilon = 0, \quad (2.30)$$

$$\delta\psi = (\Gamma^M\partial_M\Phi)\epsilon + ie^\Phi\partial_M C\Gamma^M\epsilon = 0, \quad (2.31)$$

and preserves the sixteen supersymmetries satisfying

$$\epsilon = H_7^{-1/8}\epsilon_0, \quad \gamma_{\bar{z}}\epsilon_0 = 0, \quad (2.32)$$

where  $\Psi_M$  and  $\psi$  are the ten-dimensional gravitino and dilatino respectively.

The simplest solution describes the local fields around a coincident stack of  $D7$ -branes. This local solution has a  $U(1)$  symmetry, which acts by rotations in the space transverse to the  $D7$ -branes, which is parametrized by the coordinate  $z$ . It is given by

$$\tau = i\tau_0 + \frac{g_s M}{2\pi i} \ln z, \quad (2.33)$$

so that

$$e^{-\Phi} = H_7 = \tau_0 - \frac{g_s M}{2\pi} \ln r, \quad C = \frac{g_s M}{2\pi} \theta, \quad (2.34)$$

where  $z = r e^{i\theta}$  and  $\tau_0$  is an arbitrary real constant. This solution, however, is only valid very near the branes — for small  $r$  — as  $e^{-\Phi}$  becomes negative at a finite distance and we encounter a singularity. The local solution for separated branes corresponds to

$$\tau = i\tau_0 + \frac{g_s}{2\pi i} \sum_{l=1}^M \ln(z - z_l), \quad (2.35)$$

where  $z_l$  is location of the  $l$ -th  $D7$ -brane.

As shown in [1] (see [2, 3] for more recent discussions), the local solution can be patched into global solutions that avoid the pathologies of the local one. The global solutions break the  $U(1)$  symmetry present in the local solution of coincident  $D7$ -branes. In order to describe them it is convenient to switch to the Einstein frame, where the  $SL(2, \mathbb{Z})$  invariance of Type IIB string theory is manifest. In this frame, the local metric is given by

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + dx^I dx^I + H_7 dz d\bar{z}. \quad (2.36)$$

Since  $\tau$  is defined up to the action of  $SL(2, \mathbb{Z})$  and  $Im \tau > 0$ , it follows that  $\tau$  takes values in the fundamental domain  $\mathcal{F} = \mathcal{H}^+ / SL(2, \mathbb{Z})$ , where  $\mathcal{H}^+$  is the upper half plane. In order to find a global solution for  $\tau$  one has to consider the one-to-one map  $j : \mathcal{F} \rightarrow \mathbb{C}$  from the fundamental domain  $\mathcal{F}$  to the complex plane  $\mathbb{C}$ . This map  $j$  is well-known and given by

$$j(\tau) = \frac{(\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8)^3}{\eta(\tau)^{24}}, \quad (2.37)$$

where the  $\theta$ 's are the usual theta-functions while  $\eta$  is the Dedekind  $\eta$ -function

$$\eta(\tau) = q^{1/24} \prod_n (1 - q^n), \quad (2.38)$$

where  $q = e^{2\pi i \tau}$ . Then the various solutions for  $\tau$  are given by

$$j(\tau(z)) = g(z), \quad (2.39)$$

where  $g(z)$  is an arbitrary meromorphic function in the complex plane. For a stack of  $M$  coincident  $D7$ -branes we have

$$g(z) = a + \frac{b}{z^{g_s M}}, \quad (2.40)$$

where  $a$  sets the value of the dilaton at infinity and  $b$  is related to  $\tau_0$  in (2.34). Indeed, for  $Im \tau \gg 1$ ,  $j(\tau) \simeq e^{-2\pi i \tau}$  which implies the local behavior (2.33) near  $z = 0$ .

In general, different choices of  $g(z)$  correspond to different types of  $D7$ -brane solutions. The metric can be written in the following form

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + dx^I dx^I + H_7 f \bar{f} dz d\bar{z}, \quad (2.41)$$

where as in the local case  $H_7 = e^{-\Phi}$  and where  $f$  is a holomorphic function of  $z$ . Locally, one can always choose a coordinate system where  $f \bar{f} dz d\bar{z} = dz' d\bar{z}'$  for some local coordinates

$z'$  and  $\bar{z}'$ . This brings the metric (2.41) to the usual local form (2.36). However, globally this cannot be done as discussed above. For the metric to be globally defined,  $H_7 f \bar{f}$  has to be  $SL(2, \mathbb{Z})$  invariant. The solution studied in [1] is given by

$$H_7 f \bar{f} = e^{-\Phi} \eta^2 \bar{\eta}^2 \left| \prod_{i=1}^M (z - z_i)^{-1/12} \right|^2, \quad (2.42)$$

where  $z_i$ 's are the location of the poles of  $g(z)$ , which correspond to the position of the various  $D7$ -branes in the  $z$ -plane.<sup>9</sup>

The metric (2.41) is smooth everywhere except

- 1) at  $z = z_i$  where it behaves as  $\ln |z - z_i|$  due to the presence of a  $D7$ -brane source there and
- 2) at infinity, where it has a conical singularity with deficit angle  $\delta = \frac{\pi M}{6}$ . In this paper, we will mostly be using the  $D7$ -brane background in the local form (2.28), (2.29).

However, as we explained the generalization to the global case is straightforward.

We finish this subsection by constructing the Killing spinors of the gauge theory on the  $D3$ -branes when placed in the background of the  $D7$ -branes. If we consider the  $D3/D7$  intersection in (2.1), we need the restriction of the  $D7$ -brane background to the worldvolume of the  $D3$ -branes. Then the induced metric on the  $D3$ -branes is given by:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -H_7^{-1/2} dx^+ dx^- + H_7^{1/2} dz d\bar{z}. \quad (2.43)$$

The Killing spinor equation satisfied by the four dimensional spinors  $\epsilon_\alpha^i$  that generate the worldvolume supersymmetry transformations on the  $D3$ -branes is given by<sup>10</sup>

$$D_\mu \epsilon_\alpha^i = -\frac{i}{8} e^\Phi \partial_\nu C \sigma_{\alpha\dot{\alpha}}^\nu \tilde{\sigma}_\mu^{\dot{\alpha}\beta} \epsilon_\beta^i, \quad (2.45)$$

where  $D_\mu$  is the covariant derivative in the background metric (2.43). Therefore

$$\epsilon_\alpha^i = H_7^{-1/8} \epsilon_{0\alpha}^i, \quad (2.46)$$

where

$$\tilde{\sigma}_{\bar{z}}^{\dot{\alpha}\alpha} \epsilon_\alpha^i = 0, \quad \tilde{\sigma}_+^{\dot{\alpha}\alpha} \epsilon_\alpha^i = 0, \quad (2.47)$$

thus reproducing the supersymmetry conditions derived for the brane intersection (2.8). In the next subsection we write down the action and supersymmetry transformations of the  $D3/D7$  low energy gauge theory and show that the preserved Killing spinors satisfy (2.45) subject to the constraints (2.47).

<sup>9</sup>There are restrictions on the range of  $M$  coming from the fact that for  $M$  large enough the space becomes compact. This was studied in detail in [1]. We will not discuss this point in this paper.

<sup>10</sup>To go to the four dimensional notation we have used:

$$\Gamma^\mu = i \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}. \quad (2.44)$$

### 2.3 Holographic gauge theory in background fields

In this subsection we construct the low energy gauge theory on the  $D3$ -branes when placed in the full supergravity background of the  $D7$ -branes. This is the appropriate decoupled field theory that holographically describes the physics of the dual closed string background, which we obtain in section 3 by finding the supergravity solution of the  $D3/D7$  intersection. We also construct the corresponding supersymmetry transformations and show that the action is invariant under the subset of  $\mathcal{N} = 4$  supersymmetry transformations satisfying the restrictions (2.45) and (2.47), which are precisely the supersymmetries preserved by the  $D$ -brane intersection in flat space (2.1).

There is a systematic way of constructing the action and supersymmetry transformations on a single  $D$ -brane in an arbitrary supergravity background. The starting point is to consider the covariant  $D$ -brane action in an arbitrary curved superspace background [36, 37] (which generalizes the flat space construction in [38].) These actions can in principle be expanded to all orders in the fermions around a given background, even though explicit formulas are not easy to obtain. The covariant action has  $\kappa$ -symmetry and diffeomorphism invariance. By fixing  $\kappa$ -symmetry we can gauge away sixteen of the thirty two fermions of Type IIB supergravity superspace. The remaining sixteen fermions are then identified with the gauginos filling up the SYM multiplet living on the  $D$ -brane. Likewise, worldvolume diffeomorphisms can be fixed by specifying how the brane is embedded in the background, which allows for the identification of the scalars of the SYM multiplet parametrizing the position of the  $D$ -brane.

In order to construct the explicit supersymmetry transformations of the gauge fixed action one must combine the superspace supersymmetry transformations on the physical fields together with a compensating  $\kappa$  and diffeomorphism transformation to preserve the gauge fixing condition.

Since we are interested in considering a decoupling limit, where  $\alpha' \rightarrow 0$ , this procedure simplifies considerably. In this limit the only terms in the action that survive are quadratic in the fields. Fortunately, the explicit expression for the  $D$ -brane action to quadratic order in the fermions in an arbitrary supergravity background can be found in [4]<sup>11</sup> (see also [5, 6, 39]). This approach gives the brane action quadratic in fermions with fixed  $\kappa$ -supersymmetry and diffeomorphisms in an arbitrary supersymmetric background. Therefore, we start by finding the action for a single  $D3$ -brane in the  $D7$ -brane background following [4]. Later we will show how to extend this analysis to the case when the gauge group is non-Abelian.

Let us start with the bosonic action in the  $D7$ -brane background. The action for the gauge field  $A_\mu$  is straightforward to write down. It is given by

$$S_V = -\frac{T_3}{4} \int d^4x \sqrt{-g} e^{-\Phi} F_{\mu\nu} F^{\mu\nu} - \frac{T_3}{4} \int d^4x \sqrt{-g} \partial_\mu C \epsilon^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma}, \quad (2.48)$$

---

<sup>11</sup>In that paper the action is written to quadratic order in the fermionic fields and to all order in the bosonic fields. In the decoupling limit, we will only need to extract the action to quadratic order in the bosonic fields.

where

$$T_3 = (2\pi\alpha')^2 \tau_3 = \frac{1}{2\pi g_s} = \frac{1}{g^2}, \quad (2.49)$$

where  $g$  is the SYM coupling constant. The coordinates  $x^\mu = (x^+, x^-, z, \bar{z})$  describe the coordinates along the  $D3$ -brane worldvolume as defined in (2.1). The metric used on the  $D3$ -brane worldvolume is the induced metric (2.43) from the  $D7$ -brane background.

In order to obtain the action for the scalar fields on the  $D3$ -brane it is important to properly identify which are the fields describing the  $D$ -brane fluctuations. We introduce vielbeins which are adapted to the symmetries preserved by the  $D3$ -brane  $(e^{\hat{\mu}}, e^{\hat{I}})$ , where  $\hat{\mu}$  and  $\hat{I}$  denote the flat indices along and transverse to the  $D3$ -brane respectively. The static gauge is fixed by the requirement that the pullback of the vielbein  $e^{\hat{I}}_I$  on the  $D3$ -brane vanishes and the pullback of the vielbein  $e^{\hat{\mu}}_\mu$  forms a  $D3$ -brane worldvolume vielbein. The physical scalar fields are parametrized by

$$\varphi^{\hat{I}} = e^{\hat{I}}_I \delta x^I \quad (2.50)$$

rather than by the fluctuations in the transverse coordinates  $\delta x^I$ . The scalar fields  $\varphi^{\hat{I}}$  transform under the local tangent space  $SU(4) \simeq SO(6)$  symmetries while the fluctuations  $\delta x^I$  transform under diffeomorphisms in the transverse space. This choice of the static gauge manifestly has the  $SO(6)$   $R$ -symmetry since the index  $\hat{I}$  is flat.

The low energy action for the scalar fields  $\varphi^{\hat{I}}$  can be obtained by expanding the bosonic part of the DBI action:

$$S_{\text{DBI}} = -\tau_3 \int d^4x e^{-\Phi} \sqrt{-G}. \quad (2.51)$$

$G$  is the determinant of the metric

$$G_{\mu\nu} = g_{\mu\nu} + G_{IJ} \partial_\mu \delta x^I \partial_\nu \delta x^J, \quad (2.52)$$

where  $g_{\mu\nu}$  is the induced metric (2.43) and  $G_{IJ}$  is the metric in the transverse space (2.36)

$$G_{IJ} = H^{-1/2} \delta_{IJ} = e^{\Phi/2} \delta_{IJ}, \quad (2.53)$$

where the last equality is a property of the  $D7$ -brane background.

Therefore, we find that the quadratic action for the scalar fields in the SYM multiplet is given by

$$\begin{aligned} S_{\text{Sc}} &= -\frac{T_3}{2} \int d^4x \sqrt{-g} e^{-\Phi} G_{IJ} \partial_\mu \delta x^I \partial^\mu \delta x^J \\ &= -\frac{T_3}{2} \int d^4x \sqrt{-g} e^{-\Phi} G_{IJ} \partial_\mu (e^{\hat{I}}_I \varphi^{\hat{I}}) \partial^\mu (e^{\hat{J}}_J \varphi^{\hat{J}}), \end{aligned} \quad (2.54)$$

where the worldvolume indices  $\mu$  are contracted with the induced metric (2.43) and we have used (2.50) to eliminate  $\delta x^I$  in terms of  $\varphi^{\hat{I}}$ . We note that  $\partial_\mu$  in (2.54) acts not only on  $\varphi^{\hat{I}}$  but also on the vielbein's  $e^{\hat{I}}_I$ . This fact is responsible for giving a mass to the scalar fields  $\varphi^{\hat{I}}$ . More precisely, evaluating (2.54) gives

$$S_{\text{Sc}} = -\frac{T_3}{2} \int d^4x \sqrt{-g} e^{-\Phi} (\partial_\mu \varphi^{\hat{I}} \partial^\mu \varphi^{\hat{I}} + \frac{1}{2} (\mathcal{R} + \partial^\mu \partial_\mu \Phi) \varphi^{\hat{I}} \varphi^{\hat{I}}), \quad (2.55)$$

where  $\mathcal{R}$  is the scalar curvature of the induced metric (2.43), which in terms of the dilaton field  $\Phi$  is given by:

$$\mathcal{R} = -\frac{3}{8}\partial^\mu\Phi\partial_\mu\Phi - \frac{1}{2}\partial^\mu\partial_\mu\Phi. \quad (2.56)$$

A similar mass term proportional to the curvature appears in the action of  $\mathcal{N} = 4$  Yang-Mills theory in  $\mathbb{R} \times S^3$  [7] (for a recent discussion see [8]).

For later convenience, we parametrize the six scalars  $\varphi^{\hat{I}}$  by a two-index antisymmetric tensor  $\varphi^{ij}$  of SU(4) via

$$\varphi^{\hat{I}} = \frac{1}{2}\gamma_{ij}^{\hat{I}}\varphi^{ij}, \quad \varphi^{ij} = \frac{1}{2}\tilde{\gamma}^{\hat{I}ij}\varphi^{\hat{I}}, \quad \varphi_{ij} = \frac{1}{2}\epsilon_{ijkl}\varphi^{kl}, \quad (2.57)$$

where  $\gamma_{ij}^{\hat{I}}$  are the Clebsch-Gordan coefficients that couple the **6** representation of SO(6) to the **4**'s of SU(4) labeled by the  $i, j$  indices. The Clebsch-Gordan coefficients satisfy a Clifford algebra:

$$\{\gamma^{\hat{I}}, \tilde{\gamma}^{\hat{J}}\} = 2\delta^{\hat{I}\hat{J}}. \quad (2.58)$$

In this parametrization the action of the scalar fields in the SYM multiplet is given by:

$$S_{\text{Sc}} = -\frac{T_3}{2} \int d^4x \sqrt{-g} e^{-\Phi} (\partial_\mu \varphi^{ij} \partial^\mu \varphi_{ij} + \frac{1}{2}(\mathcal{R} + \partial^\mu \partial_\mu \Phi) \varphi^{ij} \varphi_{ij}). \quad (2.59)$$

Now we move on to the action for the fermions in the SYM multiplet. As indicated earlier, the  $\kappa$ -supersymmetric DBI action depends on thirty two spinors, which can be parametrized by two ten dimensional Majorana spinors of positive chirality, denoted by  $\theta_1$  and  $\theta_2$ . Fixing  $\kappa$ -supersymmetry is equivalent to setting one of them, say  $\theta_2$  to zero. Hence, the fermionic action can be written in terms of  $\theta_1$ , which is identified with the gaugino in the SYM multiplet. The quadratic fermionic action with fixed  $\kappa$ -supersymmetry was found in [4]. Adopting their answer to our present case we obtain:

$$S_F = \frac{T_3}{2} \int d^4x \sqrt{-g} e^{-\Phi} (\bar{\theta}_1 \Gamma^\mu D_\mu \theta_1 - \bar{\theta}_1 \hat{\Gamma}_{D_3}^{-1} (\Gamma^\mu W_\mu - \Delta) \theta_1). \quad (2.60)$$

In this expression we have used:

$$\bar{\theta}_1 = i\theta_1^T \gamma^0, \quad (2.61)$$

$$\hat{\Gamma}_{D_3} = \gamma_0 \gamma_1 \gamma_2 \gamma_3, \quad (2.62)$$

$$W_\mu = \frac{1}{8} e^{-\Phi} \partial_\nu C \Gamma^\nu \Gamma_\mu, \quad (2.63)$$

$$\Delta = -\frac{1}{2} e^{-\Phi} \partial_\mu C \Gamma^\mu. \quad (2.64)$$

In order to write the action in terms of four dimensional spinors we use the basis of  $\Gamma$  matrices in (2.44) and decompose

$$\theta_1 = \begin{pmatrix} \lambda_\alpha^i \\ \bar{\lambda}_{\dot{\alpha}i} \end{pmatrix}, \quad (2.65)$$

where  $\lambda_\alpha^i$  is the four dimensional gaugino. We then obtain the action for the fermionic components of the SYM multiplet:

$$S_F = T_3 \int d^4x \sqrt{-g} e^{-\Phi} \left( \frac{i}{2} \bar{\lambda}_i \tilde{\sigma}^\mu D_\mu \lambda^i - \frac{i}{2} D_\mu \bar{\lambda}_i \tilde{\sigma}^\mu \lambda^i \right) - \frac{T_3}{4} \int d^4x \sqrt{-g} \partial_\mu C \bar{\lambda}_i \tilde{\sigma}^\mu \lambda^i. \quad (2.66)$$

In summary, the total action for the SYM multiplet in the Abelian case is then given by:

$$S_{\text{abel}} = S_V + S_{\text{Sc}} + S_F, \quad (2.67)$$

where  $S_V, S_{\text{Sc}}$  and  $S_F$  are given by (2.48), (2.59) and (2.66) respectively.

The supersymmetry transformations can be obtained from the superspace supersymmetry transformations on the physical fields with a compensating  $\kappa$  and diffeomorphism transformation to preserve the gauge fixing condition [4]. For the case under consideration we find that the action (2.67) is supersymmetric under the following transformations

$$\delta A_\mu = -i \bar{\lambda}_i \tilde{\sigma}_\mu \epsilon^i + \text{c.c.} \quad (2.68)$$

$$\delta \varphi^{ij} = (\lambda^{\alpha i} \epsilon_\alpha^j - \lambda^{\alpha j} \epsilon_\alpha^i) + \epsilon^{ijkl} \bar{\epsilon}_{\dot{\alpha} k} \bar{\lambda}^{\dot{\alpha} l} \quad (2.69)$$

$$\delta \lambda_\alpha^i = -\frac{1}{2} F_{\mu\nu} (\sigma^\mu \tilde{\sigma}^\nu)_\alpha^\beta \epsilon_\beta^i - 2i \sigma_{\alpha\dot{\alpha}}^\mu (\partial_\mu \varphi^{ij}) \bar{\epsilon}^{\dot{\alpha} j} + \frac{i}{2} \sigma_{\alpha\dot{\alpha}}^\mu (\partial_\mu \Phi) \varphi^{ij} \bar{\epsilon}^{\dot{\alpha} j}, \quad (2.70)$$

where  $\epsilon_\alpha^i$  is a Killing spinor satisfying (2.45) and subject to the constraints

$$\tilde{\sigma}_{\bar{z}}^{\dot{\alpha}\alpha} \epsilon_\alpha^i = 0, \quad \tilde{\sigma}_+^{\dot{\alpha}\alpha} \epsilon_\alpha^i = 0, \quad (2.71)$$

so that the action is invariant under eight real supersymmetries.

We note that the variation of the gaugino contains a term proportional to the derivative of the dilaton which is absent in the usual  $\mathcal{N} = 4$  SYM theory in flat space. The appearance of this term is consistent with the presence of a scalar ‘‘mass term’’ in the action (2.59). The existence of the mass term in the action indicates that a non-vanishing constant values of  $\varphi^{ij}$  does not solve equations of motion. On the other hand, the set of supersymmetric solutions can be obtained by setting the variations of the fermions to zero. Therefore, the absence of the last term in  $\delta \lambda_\alpha^i$  would indicate that any constant  $\varphi^{ij}$  was a supersymmetric solution, in direct contradiction with the equations of motion.<sup>12</sup>

The formalism using the covariant  $D$ -brane action allowed us to write a supersymmetric gauge theory action when the gauge group is Abelian. We now extend the analysis of the action and the supersymmetry transformations to the case when the gauge group is non-Abelian. The extension is relatively straightforward. In the action (2.67) we replace all derivatives  $D_\mu$  by the gauge covariant derivatives  $\mathcal{D}_\mu$ , where

$$\mathcal{D}_\mu \cdot = D_\mu \cdot + [A_\mu, \cdot], \quad (2.73)$$

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<sup>12</sup>One can perform a field redefinition and get rid of the ‘‘mass term’’ for the scalar fields in (2.58). To do this, one simply goes from  $\varphi^{\hat{I}}$  to  $\delta x^I$

$$\delta x^I = e^I_{\hat{I}} \varphi^{\hat{I}} = e^{-\frac{\Phi}{4}} \varphi^{\hat{I}} \delta_{\hat{I}}^I. \quad (2.72)$$

This transformation eliminates the ‘‘mass term’’ for the scalar fields as well as the term  $\frac{i}{2} \sigma_{\alpha\dot{\alpha}}^\mu (\partial_\mu \Phi) \varphi^{ij} \bar{\epsilon}^{\dot{\alpha} j}$  in the supersymmetry transformations for the gauginos.



replace the Chern-Simons term in (2.48) by its non-Abelian analog

$$-\frac{T_3}{4} \int d^4x \sqrt{-g} \partial_\mu C \epsilon^{\mu\nu\rho\sigma} \text{Tr}(A_\nu F_{\rho\sigma} - \frac{2}{3} A_\nu A_\rho A_\sigma), \quad (2.74)$$

and add the familiar non-Abelian couplings of  $\mathcal{N} = 4$  SYM in flat space:

$$S_{\text{nabe}} = T_3 \int d^4x \sqrt{-g} e^{-\Phi} \text{Tr}(\bar{\lambda}_{\dot{\alpha}i} [\bar{\lambda}^{\dot{\alpha}j}, \varphi^{ij}] + \lambda^{\alpha i} [\lambda_{\alpha}^j, \varphi_{ij}] - \frac{1}{2} [\varphi^{ij}, \varphi^{kl}] [\varphi_{ij}, \varphi_{kl}]). \quad (2.75)$$

In the supersymmetry transformations (2.68) we replace also all covariant derivatives  $D_\mu$  with  $\mathcal{D}_\mu$ , and add to  $\delta\lambda_\alpha^i$  the usual flat space  $\mathcal{N} = 4$  SYM commutator term  $-2[\varphi_{jk}, \varphi^{ki}] \epsilon_\alpha^j$ .

We have found the complete non-Abelian action on  $N$   $D3$ -branes when embedded in the  $D7$ -brane background. The full action is given by:

$$\begin{aligned} S = & -\frac{T_3}{4} \int d^4x \sqrt{-g} e^{-\Phi} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{T_3}{4} \int d^4x \sqrt{-g} \partial_\mu C \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left( A_\nu F_{\rho\sigma} - \frac{2}{3} A_\nu A_\rho A_\sigma \right) \\ & + T_3 \int d^4x \sqrt{-g} e^{-\Phi} \text{Tr} \left( \frac{i}{2} \bar{\lambda}_i \tilde{\sigma}^\mu \mathcal{D}_\mu \lambda^i - \frac{i}{2} \mathcal{D}_\mu \bar{\lambda}_i \tilde{\sigma}^\mu \lambda^i \right) - \frac{T_3}{4} \int d^4x \sqrt{-g} \partial_\mu C \text{Tr}(\bar{\lambda}_i \tilde{\sigma}^\mu \lambda^i) \\ & - \frac{T_3}{2} \int d^4x \sqrt{-g} e^{-\Phi} \text{Tr} \left( \mathcal{D}_\mu \varphi_{ij} \mathcal{D}^\mu \varphi^{ij} + \frac{1}{2} (\mathcal{R} + \partial^\mu \partial_\mu \Phi) \varphi^{ij} \varphi_{ij} \right) \\ & + T_3 \int d^4x \sqrt{-g} e^{-\Phi} \text{Tr} \left( \bar{\lambda}_{\dot{\alpha}i} [\bar{\lambda}^{\dot{\alpha}j}, \varphi^{ij}] + \lambda^{\alpha i} [\lambda_{\alpha}^j, \varphi_{ij}] - \frac{1}{2} [\varphi^{ij}, \varphi^{kl}] [\varphi_{ij}, \varphi_{kl}] \right). \quad (2.76) \end{aligned}$$

The action on the  $D3$ -branes (2.76) is invariant under the following explicit supersymmetry transformations

$$\begin{aligned} \delta A_\mu &= -i \bar{\lambda}_i \tilde{\sigma}_\mu \epsilon^i + \text{c.c.} \\ \delta \varphi^{ij} &= (\lambda^{\alpha i} \epsilon_\alpha^j - \lambda^{\alpha j} \epsilon_\alpha^i) + \epsilon^{ijkl} \bar{\epsilon}_{\dot{\alpha}k} \bar{\lambda}^{\dot{\alpha}l} \\ \delta \lambda_\alpha^i &= -\frac{1}{2} F_{\mu\nu} (\sigma^\mu \tilde{\sigma}^\nu)_\alpha^\beta \epsilon_\beta^i - 2i \sigma^\mu_{\alpha\dot{\alpha}} (\partial_\mu \varphi^{ij}) \bar{\epsilon}^{\dot{\alpha}j} + \frac{i}{2} \sigma^\mu_{\alpha\dot{\alpha}} (\partial_\mu \Phi) \varphi^{ij} \bar{\epsilon}^{\dot{\alpha}j} \\ & \quad - 2[\varphi_{jk}, \varphi^{ki}] \epsilon_\alpha^j, \end{aligned} \quad (2.77)$$

where  $\epsilon_\alpha^i$  is a Killing spinor satisfying (2.45) and subject to the constraints (2.71). It, thus, preserves the same eight supersymmetries preserved by the  $D$ -brane intersection.

The detailed check of the invariance of the action (2.76) under the supersymmetry transformations (2.77) is summarized in appendix B.

We finish this subsection by stating the symmetries of this field theory. The bosonic symmetry is  $ISO(1,1) \times SO(6)$ . Furthermore, the field theory is invariant under eight real supercharges. Note that the theory is not conformally invariant. The dilatations and special conformal transformations are broken by  $z$ -dependent warp-factors  $H_7(z, \bar{z})$  in (2.43).

## 2.4 The WZW surface operator

In this final subsection we show that the field theory on the  $D3/D7$  intersection describes a surface operator of  $\mathcal{N} = 4$  SYM in the  $D7$ -brane background. This surface operator,

unlike the one in [14], has a classical expression that can be written down in terms of the classical fields that appear in the Lagrangian of  $\mathcal{N} = 4$  SYM.

The strategy that we follow for determining the expression for the surface operator is to integrate out explicitly the fermions  $\chi, \bar{\chi}$  that are localized on the surface. The effect of the non-dynamical  $D7$ -brane gauge field is trivial and we suppress it in this section. In appendix *C* we show that integrating over this gauge field reproduces the same answer as when we suppress it. This same strategy was used in [16] to derive the Wilson loop operators in  $\mathcal{N} = 4$ , which were obtained by integrating out the localized degrees of freedom living on the loop arising from a brane intersection.

We want to perform the following path integral<sup>13</sup>

$$Z = e^{iS} \cdot \int [D\chi][D\bar{\chi}] \exp(iS_{\text{defect}}), \quad (2.78)$$

where:

$$S_{\text{defect}} = \int dx^+ dx^- \bar{\chi} (\partial_+ + A_+) \chi. \quad (2.79)$$

$S$  is the  $\mathcal{N} = 4$  SYM action in the  $D7$ -brane background (2.76).

We proceed to integrating out the chiral fermions localized on the surface. This is well known to produce a WZW model, which precisely captures the anomaly of the chiral fermions via the identity

$$\text{Det}(\partial_+ + A_+) = \exp(ic_R \Gamma_{\text{WZW}}(A)), \quad (2.80)$$

where  $c_R$  is the index of the representation  $R$  under which the fermions transform. The explicit expression for the WZW action one gets is

$$\begin{aligned} \Gamma_{\text{WZW}}(A) = & -\frac{1}{8\pi} \int dx^+ dx^- \text{Tr} [(U^{-1} \partial_+ U) (U^{-1} \partial_- U) - (U^{-1} \partial_+ U) (V^{-1} \partial_- V)] \\ & - \frac{1}{24\pi} \int d^3x \epsilon^{ijk} \text{Tr} [(U^{-1} \partial_i U) (U^{-1} \partial_j U) (U^{-1} \partial_k U)], \end{aligned} \quad (2.81)$$

where  $U$  and  $V$  are  $U(N)$  group elements nonlocally related to the gauge field of  $\mathcal{N} = 4$  SYM:

$$A_+ = U^{-1} \partial_+ U \quad A_- = V^{-1} \partial_- V. \quad (2.82)$$

We note that  $\Gamma_{\text{WZW}}(A)$  differs from the conventional WZW model action by the addition of a local counterterm:

$$\frac{1}{8\pi} \int dx^+ dx^- \text{Tr} [(U^{-1} \partial_+ U) (V^{-1} \partial_- V)] = \frac{1}{8} \int dx^+ dx^- \text{Tr} A_+ A_-. \quad (2.83)$$

The addition of this term is needed to guarantee that (2.81) reproduces the correct chiral anomaly. Indeed, it is straightforward to show that under a  $U(N)$  gauge transformation  $\delta A_\mu = \partial_\mu + [A_\mu, L]$  we have that the WZW action (2.81) is not invariant:

$$\delta \Gamma_{\text{WZW}}(A) = \frac{1}{8\pi} \int dx^+ dx^- \text{Tr} [L (\partial_+ A_- - \partial_- A_+)]. \quad (2.84)$$

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<sup>13</sup>After this integral is performed, we must still integrate over the  $\mathcal{N} = 4$  SYM fields.

This gives the same anomalous variation as the usual anomaly in two dimensions (2.12). We recall that our complete action, which combines the  $\mathcal{N} = 4$  SYM action on the  $D7$ -brane background (2.76) with the defect term in (2.79) is not anomalous. The anomaly produced by the WZW action is precisely cancelled by a Chern-Simons term.

We also note that  $\Gamma_{\text{WZW}}(A)$  is not invariant under the supersymmetry transformations (2.77), unlike the original action  $S_{\text{defect}}$ . But we recall that the Chern-Simons terms always has a boundary term under any variation of the gauge field and that this boundary contribution cancels the variation of  $\Gamma_{\text{WZW}}(A)$  proportional to  $\delta A_-$ . For this cancellation to occur, it is also crucial to add the local counterterm (2.83).

Therefore, integrating out the localized fields has the effect of inserting the following surface operator into the gauge theory action (2.76):

$$\mathcal{O}_\Sigma = \exp(iM\Gamma_{\text{WZW}}(A)). \tag{2.85}$$

The surface operator is described by a  $U(N)$  WZW model at level  $M$ . The explicit form of the action is (2.81), where  $U$  and  $V$  are  $U(N)$  valued group elements that are nonlocally related to the  $\mathcal{N} = 4$  SYM gauge fields via:

$$A_+ = U^{-1}\partial_+U \quad A_- = V^{-1}\partial_-V. \tag{2.86}$$

The surface operator (2.85) is supersymmetric under the transformations (2.77) and  $U(N)$  invariant when combined with the gauge theory action in the  $D7$ -brane background (2.76).

Using the explicit expression for the surface operator one can study its properties in perturbation theory. For the case when  $\Sigma = \mathbb{R}^{1,1}$  we expect that supersymmetry requires  $\langle \mathcal{O}_\Sigma \rangle = 1$ , just like in the case of the Wilson line. Another interesting case to consider — which is related by a conformal transformation to the Euclidean version of the previous case — is when  $\Sigma = S^2$ . In this case  $\Sigma$  is curved and we expect that there is a conformal anomaly associated with the surface which would be interesting to compute explicitly. The bulk description discussed in the next section supports these expectations, as we find that at least in the probe approximation that  $\langle \mathcal{O}_\Sigma \rangle = 1$  and that there is a conformal anomaly for the cases  $\Sigma = \mathbb{R}^{1,1}$  and  $S^2$  respectively.

Given that these operators are supersymmetric one may expect that the computation of their expectation value is captured by a simpler model, similar to what happens for circular Wilson loops [31, 32]. One may be able to derive the reduced model by topologically twisting the gauge theory by the supercharges preserved by the surface operator.

### 3. The bulk description

In this section we study the physics of the surface operator from a dual gravitational point of view. We find that there is a regime in which the  $D7$ -branes can be treated as a probe brane in  $AdS_5 \times S^5$  and identify the corresponding regime in the gauge theory. We also find the exact solutions of the Type IIB supergravity equations of motion — which take the backreaction of the  $D7$ -branes into account — which are dual to the surface operators in the gauge theory we have constructed in this paper.

### 3.1 The probe approximation and anomaly suppression

In the previous section we have constructed the decoupled low energy effective field theory living on the  $D3/D7$  intersection (2.1). Following [40] our aim in this section is to provide the bulk gravitational description of this field theory. This requires finding the supergravity solution describing the brane intersection (2.1) [40].

In the absence of the  $D7$ -branes, the gauge theory on  $N$   $D3$ -branes is dual to string theory in  $AdS_5 \times S^5$  [40]. We are interested in understanding what the effect of introducing the  $D7$ -branes is in the bulk description.

One may try to first consider the  $D7$ -branes as a small perturbation around the  $AdS_5 \times S^5$  background. The parameter that controls the gravitational backreaction due to the  $M$   $D7$ -branes can be extracted from the supergravity equations of motion. It is governed by

$$\epsilon = M \cdot G_{10} \tau_7 = g_s M = \frac{g^2}{2\pi} M. \tag{3.1}$$

In the last step we have written the parameter using gauge theory variables, where  $g$  is the gauge theory coupling constant. Therefore, we can treat the  $D7$ -branes as probes in  $AdS_5 \times S^5$  as long as  $g^2 M$  is small.

In the regime where  $g^2 M$  is small we can consistently treat the  $D7$ -branes in the probe approximation. It is straightforward to show that the  $D7$ -brane equations of motion are solved by the embedding (2.1) even when we place the  $D7$ -branes in the non-trivial supergravity background produced by the  $D3$ -branes. Upon taking the  $D3$ -brane near horizon limit, the brane embedding geometry is that of  $AdS_3 \times S^5$  [41].

We are now in a position to determine what is the field theory counterpart of the bulk probe approximation. We recall that the gauge theory we constructed in the previous section is defined on the  $D7$ -brane background. In the probe regime, where  $g^2 M \ll 1$ , the background produced by the  $D7$ -branes becomes trivial, as the metric becomes flat, the dilaton goes to a constant and the RR flux vanishes. Hence, in this limit we get the following gauge theory

$$S = S_{\mathcal{N}=4} + \int dx^+ dx^- \bar{\chi}(\partial_+ + A_+) \chi, \tag{3.2}$$

where  $S_{\mathcal{N}=4}$  is the standard action of  $\mathcal{N} = 4$  SYM in flat space.

However, we have argued that it was crucial to consider the gauge theory on the full  $D7$ -brane geometry, so as to get an anomaly free and supersymmetric theory. The resolution lies in the observation that the gauge anomaly is suppressed in this limit. In order to better understand the parameter controlling the anomaly, it is convenient to rescale the gauge fields in the action as follows  $A_\mu \rightarrow g A_\mu$ . In this presentation it becomes clear what the effect of the coupling constant is on physical quantities. The quantum effective action obtained by integrating the fermions is anomalous, the obstruction to gauge invariance being measured by<sup>14</sup>

$$\delta_L S = \frac{g^2 M}{8\pi} \int dx^+ dx^- \text{Tr}_{U(N)}(LdA), \tag{3.3}$$

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<sup>14</sup>In the frame where the coupling constant controls the interaction vertices in gauge theory, the gauge parameter must also be rescaled  $L \rightarrow gL$ .

so that the anomaly is controlled by the same parameter that controls the backreaction of the  $D7$ -branes in the bulk (3.1), and is therefore suppressed in the probe limit  $g^2M \rightarrow 0$ .

Note that to leading order in the  $g^2M$  expansion the two dimensional Poincare symmetry of the gauge theory is enlarged to  $SO(2,2) \simeq SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ , as long as the  $D7$ -branes are coincident. This can be understood from the point of view of the symmetries of  $\mathcal{N} = 4$  SYM in flat space. A surface  $\Sigma = \mathbb{R}^{1,1} \subset \mathbb{R}^{1,3}$  is invariant under an  $SO(2,2)$  subgroup of the  $SO(2,4)$  four dimensional conformal group. The symmetries are generated by  $P_\mu, M_{\mu\nu}, K_\mu$  and  $D$ , with  $\mu = 0, 1$ , where  $K_\mu$  and  $D$  generate the special conformal and dilatation transformations respectively. In this case — where the  $D7$ -branes are coincident — the theory acquires eight extra supersymmetries, which correspond to conformal supersymmetries. Indeed, the  $S_{\mathcal{N}=4}$  term in (3.2) is invariant under sixteen superconformal supersymmetries generated by  $\varepsilon_\alpha^i$ . The second term in (3.2), given by  $S_{\text{defect}}$ , is invariant under the conformal supersymmetries generated by:

$$\tilde{\sigma}_-^{\dot{\alpha}\alpha} \varepsilon_\alpha^i = 0. \tag{3.4}$$

To see this consider the relevant superconformal transformations

$$\delta A_\mu = -ix^\nu \lambda^{\beta i} \sigma_{\mu\beta\dot{\alpha}} \tilde{\sigma}_\nu^{\dot{\alpha}\alpha} \varepsilon_\alpha^i + \text{c.c.}, \quad \delta\chi = 0. \tag{3.5}$$

It is straightforward to show that  $\delta A_+ = 0$  if (3.4) is fulfilled and the defect action is localized at  $z = 0$ . All these symmetries combine into the  $SU(1,1|4) \times SL(2, \mathbb{R})$  supergroup [42].

Once  $g^2M$  corrections are taken into account, so that the anomaly, the Chern-Simons terms and the  $D7$ -brane background cannot be neglected, the symmetries are broken down<sup>15</sup> to  $ISO(1,1) \times SO(6)$  and the theory is invariant under eight supersymmetries. Even if we start with coincident  $D7$ -branes, once one takes into account the proper global solution (2.42), the  $U(1)$  symmetry is broken.

Let's now consider the symmetries of the bulk theory in the probe approximation. When the  $M$   $D7$ -branes are coincident the  $D7$ -branes are invariant under  $SO(2,2) \times SO(2) \times SO(6)$ . The  $SO(2,2)$  and  $SO(6)$  symmetries act by isometries on the  $AdS_3$  and  $S^5$  worldvolume geometry respectively. The  $U(1)$  symmetry rotates the  $z$ -plane in (2.1). We show (see appendix D) that the coincident  $D7$ -branes also preserve half of the Type IIB supersymmetries, which coincide precisely with the Poincare and special conformal supersymmetries preserved in the gauge theory, which are given by

$$\tilde{\sigma}_+^{\dot{\alpha}\alpha} \varepsilon_\alpha^i = 0 \tag{3.6}$$

and

$$\tilde{\sigma}_-^{\dot{\alpha}\alpha} \varepsilon_\alpha^i = 0 \tag{3.7}$$

respectively. The unbroken symmetries combine to form a chiral superconformal group, which is an  $SU(1,1|4) \times SL(2, \mathbb{R})$  supergroup, as thus coincides with the gauge theory

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<sup>15</sup>This is similar to the breaking of conformal invariance by  $g^2M$  effects that occurs when considering  $\mathcal{N} = 4$  SYM coupled to  $M$  hypermultiplets, where the  $\beta$ -function is proportional to  $g^2M$ , so that  $g^2M$  effects break conformal symmetry.

symmetries discussed above. If the  $D7$ -branes are not coincident, in both field and gravity theory the symmetry is broken down to  $ISO(1,1) \times SO(6)$  and only eight supersymmetries survive.

The  $AdS_3 \times S^5$   $D7$ -brane ends on the surface  $\Sigma$  on the boundary of  $AdS_5 \times S^5$ , thus providing boundary conditions for the surface operator. One can use the probe  $D7$ -brane to calculate the expectation value of the surface operator in the probe regime. In the semiclassical approximation it is given by [43, 44]

$$\langle \mathcal{O}_\Sigma \rangle = \exp(-S_{D7}^{on-shell}). \tag{3.8}$$

For the brane embedding at hand the  $D7$ -brane on-shell action is given by

$$S_{D7}^{on-shell} = \tau_7 L^8 \text{vol}(S^5) \text{vol}_{\text{ren}}(AdS_3), \tag{3.9}$$

where  $L$  is the  $AdS_5/S^5$  radius,  $\text{vol}(S^5)$  is the volume of the  $S^5$  and  $\text{vol}_{\text{ren}}(AdS_3)$  is the renormalized volume of  $AdS_3$ . As usual the bulk action is infrared divergent and requires renormalization. This is accomplished by adding covariant counterterms. It is easy to show that the renormalized volume of  $AdS_3$  vanishes so we find that  $\langle \mathcal{O}_\Sigma \rangle = 1$  in the probe approximation. The same answer is obtained for the gauge theory in the probe approximation (3.2), as one just gets the partition function over free fermions.

One may consider surface operators defined on surfaces  $\Sigma$  other than  $\mathbb{R}^{1,1}$  in the probe approximation. In the bulk, this corresponds to considering  $D7$ -brane solutions of the DBI equations of motion that end on the boundary of  $AdS_5 \times S^5$  on  $\Sigma$ . The case when  $\Sigma = S^2$  can be obtained easily from the euclidean solution with  $\Sigma = \mathbb{R}^2$  by acting with a broken special conformal transformation. In this case the bulk  $D7$  brane is still  $AdS_3 \times S^5$ , but now  $AdS_3$  is in global coordinates and the brane ends on the boundary of  $AdS_5$  on an  $S^2$ . In this case, the calculation of  $D$ -brane action is non-trivial as the renormalized volume of global  $AdS_3$  is non-trivial. In this case, one finds that the  $D7$ -brane has a conformal anomaly, similar to the one discussed in [46, 45] in the context of  $M2$ -branes ending on an  $S^2$  in  $AdS_7 \times S^4$ . This is encoded in the coefficient of the logarithmic divergence of the on-shell action, which for  $M$   $D7$ -branes is controlled by  $\tau_7 L^8 = g^2 M N^2$ . It would be interesting to calculate the corresponding conformal anomaly in the gauge theory in perturbation theory.

### 3.2 The supergravity solution

In this final subsection we find the bulk description of a surface operator in terms of solution of the Type IIB supergravity. According to AdS/CFT duality, this solution is obtained by taking the near-horizon limit of the supergravity solution of the brane intersection (2.1). The explicit form of the solution corresponding to (2.1) is given by:

$$\begin{aligned} ds^2 &= -H_3^{-1/2} H_7^{-1/2} dx^+ dx^- + H_3^{-1/2} H_7^{1/2} dz d\bar{z} + H_3^{1/2} H_7^{-1/2} dx^I dx^I \\ e^{-\Phi} &= H_7 \\ F_{0123I} &= H_7 \partial_I H_3^{-1} \\ \partial_{\bar{z}} \tau &= 0 \quad \text{where} \quad \tau = C + i e^{-\Phi}. \end{aligned} \tag{3.10}$$

$H_3 = H_3(x^I)$  is an arbitrary harmonic function in the space transverse to the  $D3$ -branes while  $H_7 = H_7(z, \bar{z})$  determines the  $D7$ -brane contribution and it is of the same form as in section 2. It is straightforward to show that this supergravity background solves the Type IIB supergravity Killing spinor equations and that moreover the space of solutions is eight real dimensional and can be parametrized by four dimensional spinors satisfying the constraints (2.5) and (2.6).

Here we are interested in the supergravity solution describing the decoupled gauge theory constructed earlier and that lives on the  $D3/D7$  intersection. This corresponds to taking the near horizon limit of the supergravity solution corresponding to the case when the  $N$   $D3$ -branes are coincident — so that  $H = 1 + L^4/\rho^4$  — where  $dx^I dx^I = d\rho^2 + \rho^2 d\Omega_5$ . In this limit the metric can be written in terms of an  $AdS_3 \times S^5$  factor. The geometry describing the surface operator is given by

$$ds^2 = H_7^{-1/2} (ds_{AdS_3}^2 + L^2 d\Omega_5) + \frac{\rho^2}{L^2} H_7^{1/2} dz d\bar{z}, \tag{3.11}$$

where:

$$ds_{AdS_3}^2 = -\frac{\rho^2}{L^2} dx^+ dx^- + L^2 \frac{d\rho^2}{\rho^2}. \tag{3.12}$$

This metric reveals several interesting features of the holographic correspondence. We have argued that it is inconsistent to describe a low energy field theory with anomaly inflow by treating the gauge theory in flat space. We have argued that the proper description of the system is in terms of the gauge theory in the supergravity background produced by the other brane. In particular, for our intersection, we have constructed the gauge theory on the  $D3$ -branes in the background of the  $D7$ -branes and found that this field theory has all the expected properties. We can now use the dual supergravity solution (3.12) to indeed infer that the holographic dual gauge theory lives in the background geometry of the  $D7$ -branes and not in flat space. Indeed if we analyze the metric living on the conformal boundary — where  $\rho \rightarrow \infty$  — we precisely get the metric on which the gauge theory lives (2.43).

The solution also gives information about the non-perturbative behavior of the symmetries of the gauge theory. As we discussed earlier in this section, the gauge theory has  $SO(2,2)$  symmetry to leading order in a  $g^2 M$  expansion. This symmetry is intimately related to the geometrical surface on which the fermions live. However, once the  $g^2 M$  corrections are turned on and the  $D7$ -brane backreaction cannot be neglected, the conformal symmetry is broken. The dual geometry (3.11) has the same symmetries. In particular, the  $SO(2,2)$  symmetry is broken down to  $ISO(1,1)$ . First, the warp-factor  $H_7$  is not invariant under dilatations and special conformal transformations just like in field theory. Second, we see that the  $AdS_3$  radial coordinate  $\rho$  does not decouple from the transverse space and appears explicitly in the transverse metric. As usual, the  $SO(2,2)$  conformal transformations correspond to  $AdS_3$  isometries. However, since  $AdS_3$  isometries act non-trivially on  $\rho$  and  $z$  we find that the  $SO(2,2)$  conformal symmetry of the surface operator is broken down to  $ISO(1,1)$ . The supersymmetries are also reduced with respect to the probe approximation. This can be shown (see appendix E) by explicitly solving the Type

IIB Killing spinor equations in the background (3.11). The explicit Killing spinor is given by

$$\epsilon = h(\theta, \varphi_a) H_7^{-1/2} \rho^{1/2} \epsilon_0, \quad (3.13)$$

where  $h(\theta, \varphi_a)$  is the standard contribution from  $S^5$  [9, 10] (see appendix D for the explicit expression). In addition,  $\epsilon$  (as well as  $\epsilon_0$ ) is subject to the constraints (2.5) and (2.6), which give rise to eight real supersymmetries. Thus, we obtain the same symmetries as those preserved by the gauge theory.

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## A. The $\sigma$ -matrix conventions

The  $\sigma$ -matrices  $\sigma_\mu^{\alpha\dot{\alpha}}$  are defined in the usual way:<sup>16</sup>

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A.1)$$

In addition, we define:

$$\tilde{\sigma}_\mu^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\beta} \epsilon^{\alpha\beta} \sigma_{\mu\beta\dot{\beta}}, \quad (A.2)$$

$$\tilde{\sigma}_\mu = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3). \quad (A.3)$$

These matrices satisfy the following properties:

$$(\sigma_\mu \tilde{\sigma}_\nu + \sigma_\nu \tilde{\sigma}_\mu)_\alpha^\beta = -2\eta_{\mu\nu} \delta_\alpha^\beta, \quad (A.4)$$

$$(\tilde{\sigma}_\mu \sigma_\nu + \tilde{\sigma}_\nu \sigma_\mu)^{\dot{\alpha}\dot{\beta}} = -2\eta_{\mu\nu} \delta^{\dot{\alpha}\dot{\beta}}, \quad (A.5)$$

$$\text{tr}(\sigma_\mu \tilde{\sigma}_\nu) = -2\eta_{\mu\nu}, \quad (A.6)$$

$$\sigma^\mu_{\alpha\dot{\alpha}} \tilde{\sigma}_\mu^{\dot{\beta}\beta} = -2\delta_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}}, \quad (A.7)$$

$$\sigma_\mu \tilde{\sigma}_\nu \sigma_\rho = (\eta_{\mu\rho} \sigma_\nu - \eta_{\nu\rho} \sigma_\mu - \eta_{\mu\nu} \sigma_\rho) + i\epsilon_{\mu\nu\rho\sigma} \sigma^\sigma, \quad (A.8)$$

$$\tilde{\sigma}_\mu \sigma_\nu \tilde{\sigma}_\rho = (\eta_{\mu\rho} \tilde{\sigma}_\nu - \eta_{\nu\rho} \tilde{\sigma}_\mu - \eta_{\mu\nu} \tilde{\sigma}_\rho) - i\epsilon_{\mu\nu\rho\sigma} \tilde{\sigma}^\sigma. \quad (A.9)$$

In the paper we go from coordinates  $x^\mu$  to:

$$x^\pm = x^0 \pm x^1, \quad z = x^2 + ix^3. \quad (A.10)$$

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<sup>16</sup>In this appendix the index  $\mu$  is assumed to be flat. In curved space-time we will have to replace in all expressions  $\eta_{\mu\nu}$  by the space-time metric.



In this basis, we obtain:

$$\sigma^- = \sigma^0 - \sigma^1 = -\sigma_0 - \sigma_1 = -\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (\text{A.11})$$

$$\tilde{\sigma}^- = \tilde{\sigma}^0 - \tilde{\sigma}^1 = -\tilde{\sigma}_0 - \tilde{\sigma}_1 = -\sigma_0 + \sigma_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (\text{A.12})$$

$$\text{etc.} \quad (\text{A.13})$$

In particular, we have

$$\tilde{\sigma}_+ = \eta_{+-}\tilde{\sigma}^- = -\frac{1}{2}\tilde{\sigma}^- = \frac{1}{2}(\tilde{\sigma}_0 + \tilde{\sigma}_1) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (\text{A.14})$$

and:

$$\tilde{\sigma}_{\bar{z}} = \eta_{\bar{z}z}\tilde{\sigma}^z = \frac{1}{2}(\tilde{\sigma}_2 + i\tilde{\sigma}_3) = \frac{1}{2}\begin{pmatrix} -i & i \\ -i & i \end{pmatrix}. \quad (\text{A.15})$$

The restriction on the supersymmetry parameter (2.8) found in the paper can be written as:

$$\tilde{\sigma}_+^{\dot{\alpha}\alpha}\epsilon_{\alpha}^i = 0 \quad \text{or} \quad \tilde{\sigma}_{\bar{z}}^{\dot{\alpha}\alpha}\epsilon_{\alpha}^i = 0. \quad (\text{A.16})$$

Both equations in (A.16) imply that  $\epsilon_1^i = \epsilon_2^i$ .

## B. Explicit check of the supersymmetry of the action

In this appendix, we explicitly show that the non-Abelian  $D3$ -brane action in the  $D7$ -brane background given in (2.76) is invariant under the supersymmetry transformations in (2.77). The action has the following structure

$$S = S_V + S_{\text{Sc}} + S_F + S_{\text{nab}}, \quad (\text{B.1})$$

where:

$$\begin{aligned} S_V &= -\frac{T_3}{4} \int d^4x \sqrt{-g} e^{-\Phi} \text{Tr} F_{\mu\nu} F^{\mu\nu} - \frac{T_3}{4} \int d^4x \sqrt{-g} \partial_{\mu} C e^{\mu\nu\rho\sigma} \text{Tr} \left( A_{\nu} F_{\rho\sigma} - \frac{2}{3} A_{\nu} A_{\rho} A_{\sigma} \right), \\ S_{\text{Sc}} &= -\frac{T_3}{2} \int d^4x \sqrt{-g} e^{-\Phi} \text{Tr} \left( \mathcal{D}_{\mu} \varphi^{ij} \mathcal{D}^{\mu} \varphi_{ij} + \frac{1}{2} (\mathcal{R} + \partial^{\mu} \partial_{\mu} \Phi) \varphi^{ij} \varphi_{ij} \right), \\ S_F &= T_3 \int d^4x \sqrt{-g} e^{-\Phi} \text{Tr} \left( \frac{i}{2} \bar{\lambda}_i \tilde{\sigma}^{\mu} \mathcal{D}_{\mu} \lambda^i - \frac{i}{2} \mathcal{D}_{\mu} \bar{\lambda}_i \tilde{\sigma}^{\mu} \lambda^i \right) - \frac{T_3}{4} \int d^4x \sqrt{-g} \partial_{\mu} C \bar{\lambda}_i \tilde{\sigma}^{\mu} \lambda^i, \\ S_{\text{nab}} &= T_3 \int d^4x \sqrt{-g} e^{-\Phi} \text{Tr} \left( \bar{\lambda}_{\dot{\alpha}i} [\bar{\lambda}^{\dot{\alpha}}_j, \varphi^{ij}] + \lambda^{\alpha i} [\lambda_{\alpha}^j, \varphi_{ij}] - \frac{1}{2} [\varphi^{ij}, \varphi^{kl}] [\varphi_{ij}, \varphi_{kl}] \right). \quad (\text{B.2}) \end{aligned}$$

In looking at the supersymmetry variation of the action we do not write the terms that cancel exactly in the same way as they cancel in  $\mathcal{N} = 4$  SYM theory in flat space. That is, we only keep the terms which contain derivatives of the background supergravity fields

and  $\epsilon^i$  and discuss how they cancel. Let us first look at the variation of the terms in the action involving the gauge fields  $S_V$ . We obtain:

$$\begin{aligned} \delta S_V = & -\frac{T_3}{2} \int d^4x \sqrt{-g} \partial_\mu \tau \text{Tr} \left( F^{\mu\nu} - \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \right) (\bar{\lambda}_i \tilde{\sigma}_\nu \epsilon^i) \\ & + \frac{T_3}{2} \int d^4x \sqrt{-g} \partial_\mu \bar{\tau} \text{Tr} \left( F^{\mu\nu} + \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \right) (\bar{\lambda}_i \tilde{\sigma}_\nu \epsilon^i) + \text{c.c.} \end{aligned} \quad (\text{B.3})$$

Using the fact that  $\tau$  is holomorphic and that  $\epsilon^i$  satisfies equations (A.16), it is easy to see that the first term in the above expression vanishes and only the second term containing  $\partial_\mu \bar{\tau}$  survives. Now we vary the fermionic terms in the action in  $S_F$  under:

$$\delta \lambda_\alpha^i = -\frac{1}{2} F_{\mu\nu} (\sigma^\mu \tilde{\sigma}^\nu)_\alpha^\beta \epsilon_\beta^i. \quad (\text{B.4})$$

By using the background Killing spinor equation (2.45), we find that the terms in  $\delta S_F$  with  $\partial_\mu \tau$  cancel and terms with  $\partial_\mu \bar{\tau}$  produce exactly the same expression as in (B.3) but with the opposite sign. This provides the cancellation of terms involving the vectors fields and the fermions.

Now we consider the variation in the action  $S_{\text{Sc}}$  involving the scalars. It is straightforward to obtain that:

$$\begin{aligned} \delta S_{\text{Sc}} = & 2T_3 \int d^4x \sqrt{-g} (\partial^\mu e^{-\Phi}) \text{Tr} (\mathcal{D}_\mu \varphi_{ij}) (\lambda^i \epsilon^j) \\ & - T_3 \int d^4x \sqrt{-g} e^{-\Phi} (\mathcal{R} + \partial^\mu \partial_\mu \Phi) \text{Tr} (\varphi_{ij} (\lambda^i \epsilon^j)) + \text{c.c.} \end{aligned} \quad (\text{B.5})$$

These terms cancel against the variation of  $S_F$  under:

$$\delta \bar{\lambda}_{\dot{\alpha}i} = 2i \epsilon^{\alpha j} \sigma_{\alpha\dot{\alpha}}^\mu \mathcal{D}_\mu \varphi_{ij} - \frac{i}{2} \epsilon^{\alpha j} \sigma_{\alpha\dot{\alpha}}^\mu (\partial_\mu \Phi) \varphi_{ij}. \quad (\text{B.6})$$

Let us make some remarks on how the terms containing derivatives of  $C$  cancel when we vary  $S_F$  (such terms are not present in the variation of  $S_{\text{Sc}}$  in (B.5)). Consider the variation of the second term in  $S_F$  under (B.6). We get:

$$\begin{aligned} & -\frac{i}{2} \int d^4x \sqrt{-g} \partial_\mu C \text{Tr} (\mathcal{D}_\nu \varphi_{ij} (\lambda^i \sigma^\mu \tilde{\sigma}^\nu \epsilon^j)) \\ & + \frac{i}{8} \int d^4x \sqrt{-g} \partial_\mu C \partial_\nu \Phi \text{Tr} (\varphi_{ij} (\lambda^i \sigma^\mu \tilde{\sigma}^\nu \epsilon^j)) + \text{c.c.} \end{aligned} \quad (\text{B.7})$$

In both terms we anticommute  $\sigma^\mu$  and  $\tilde{\sigma}^\nu$  using (A.4). Then each term in (B.7) will split into two terms. The first two terms yield:

$$i \int d^4x \sqrt{-g} \partial_\mu C \text{Tr} (\mathcal{D}^\mu \varphi_{ij} (\lambda^i \epsilon^j)) - \frac{i}{4} \int d^4x \sqrt{-g} \partial_\mu C \partial^\mu \Phi \text{Tr} (\varphi_{ij} (\lambda^i \epsilon^j)) + \text{c.c.} \quad (\text{B.8})$$

They cancel against the variation of the fermion kinetic term when we rewrite  $D\epsilon$  in terms of the derivative of the axion by using (2.45). The remaining two terms are:

$$\begin{aligned} & \frac{i}{2} \int d^4x \sqrt{-g} \partial_\mu C \text{Tr} (\mathcal{D}_\nu \varphi_{ij} (\lambda^i \sigma^\nu \tilde{\sigma}^\mu \epsilon^j)) \\ & - \frac{i}{8} \int d^4x \sqrt{-g} \partial_\mu C \partial_\nu \Phi \text{Tr} (\varphi_{ij} (\lambda^i \sigma^\nu \tilde{\sigma}^\mu \epsilon^j)) + \text{c.c.} \end{aligned} \quad (\text{B.9})$$

We now use the condition that  $\tau$  is a holomorphic function together with the projection satisfied by the Killing spinor  $\tilde{\sigma}_{\bar{z}}\epsilon^i = 0$ . This can be summarized by:

$$\partial_{\mu}\tau \tilde{\sigma}^{\mu}\epsilon^i = 0. \quad (\text{B.10})$$

Using this equation, we can get rid of the terms with derivatives of  $C$  in (B.9) and write them using derivatives of  $e^{-\Phi}$ . The cancellation of such terms arising in  $\delta S_F$  and  $\delta S_{\text{Sc}}$  is already straightforward.

In the last step, we vary  $S_F$  under the remaining term in the variation of  $\lambda$ :

$$\delta\lambda_{\alpha}^i = -2[\varphi_{jk}, \varphi^{ki}]\epsilon_{\alpha}^j. \quad (\text{B.11})$$

The terms containing  $\partial C$  cancel (after we use the Killing spinor equation  $D\epsilon \sim \partial C$  as in (2.45)) and we obtain:

$$-i \int d^4x \sqrt{-g} (\partial_{\mu} e^{-\Phi}) \text{Tr} \left( [\varphi_{jk}, \varphi^{ki}] (\bar{\lambda}_i \tilde{\sigma}^{\mu} \epsilon^j) \right) + \text{c.c.} \quad (\text{B.12})$$

This term cancels against the variation of  $S_{\text{nab}}$ . In varying  $S_{\text{nab}}$  we only have to consider:

$$\delta\bar{\lambda}_{\dot{\alpha}i} = -\frac{i}{2}\epsilon^{\alpha j}\sigma^{\mu}_{\alpha\dot{\alpha}}(\partial_{\mu}\Phi)\varphi_{kj}. \quad (\text{B.13})$$

Anything else gives terms which cancel just like in flat background. It is straightforward to see that the variation of  $S_{\text{nab}}$  under (B.13) indeed cancels (B.12). This finishes our proof of the supersymmetry of the action.

## C. Integrating out the defect fields

In this appendix, we perform the explicit integration over the defect fields. We split the  $U(N)$  gauge field into an  $SU(N)$  gauge field which we denote by  $A$  and a  $U(1)$  gauge field which we denote by  $a$ . Similarly the  $U(M)$  gauge field is decomposed into an  $SU(M)$  gauge field  $\tilde{A}$  and a  $U(1)$  gauge field  $\tilde{a}$ . Therefore, we want to perform the following path integral

$$Z = \int [D\chi][D\bar{\chi}][D\tilde{A}][D\tilde{a}] \exp \left[ \left( (S_{\text{defect}} + S_{\text{CS}}(\tilde{A}) + S_{\text{CS}}(\tilde{a}) + S_{\text{CS}}(a, \tilde{a})) \right) \right], \quad (\text{C.1})$$

where:

$$S_{\text{defect}} = \int dx^+ dx^- \bar{\chi} \left( \partial_+ + A_+ + \tilde{A}_+ + a_+ - \tilde{a}_+ \right) \chi. \quad (\text{C.2})$$

Here we took into account that  $\chi$  carries the opposite  $U(1)$  charges under  $U(N)$  and  $U(M)$  action. The non-Abelian Chern-Simons term  $S_{\text{CS}}(\tilde{A})$  is given by

$$S_{\text{CS}}(\tilde{A}) = -\frac{(2\pi\alpha')^2\tau_7}{2} \int G_5 \wedge \text{Tr} \left( \tilde{A} \wedge d\tilde{A} + \frac{2}{3}\tilde{A} \wedge \tilde{A} \wedge \tilde{A} \right). \quad (\text{C.3})$$

Similarly:

$$S_{\text{CS}}(\tilde{a}) = -\frac{(2\pi\alpha')^2\tau_7}{2} \int G_5 \wedge \tilde{a} \wedge d\tilde{a}. \quad (\text{C.4})$$

Finally, the mixed Chern-Simons terms are given by

$$S_{\text{CS}}(a, \tilde{a}) = -\frac{(2\pi\alpha')^2 \tau_3}{2} N \int G_1 \wedge a \wedge \tilde{f} + \frac{(2\pi\alpha')^2}{2} M \int G_5 \wedge \tilde{a} \wedge f, \quad (\text{C.5})$$

where  $f = da$  and  $\tilde{f} = d\tilde{a}$ .

Integrating the fermions in (C.1) yields

$$\int [D\chi][D\bar{\chi}] \exp(iS_{\text{defect}}) = \exp \left[ i \left( M\Gamma_{\text{WZW}}(A) + N\Gamma_{\text{WZW}}(\tilde{A}) + NM\Gamma_{\text{WZW}}(a, \tilde{a}) \right) \right]. \quad (\text{C.6})$$

We must now integrate the D7-brane gauge fields  $\tilde{A}$  and  $\tilde{a}$  in (C.1). The gauge field  $\tilde{A}$  is completely decoupled from the  $\mathcal{N} = 4$  SYM gauge fields  $A$  and  $a$ . Therefore the integral over  $\tilde{A}$ , which appears in the action through the terms  $N\Gamma_{\text{WZW}}(\tilde{A}) + S_{\text{CS}}(\tilde{A})$  just gives a constant.

Now we have to perform the integral over  $\tilde{a}$ . In order to simplify the formulas, we consider the case of  $M$  coincident  $D7$ -branes with the local  $U(1)$  symmetry. In this case the RR one-form flux is given by:

$$G_1 = \frac{g_s M}{2\pi} d\theta. \quad (\text{C.7})$$

A similar analysis can be easily generalized for the global solutions, as all we require is that  $G_1$  satisfies the Bianchi identities. The path integral we have to study is

$$\int [D\tilde{a}] \exp(i\Gamma(a, \tilde{a})), \quad (\text{C.8})$$

where

$$\Gamma(a, \tilde{a}) = NM\Gamma_{\text{WZW}}(a - \tilde{a}) + S_{\text{CS}}(\tilde{a}) + S_{\text{CS}}(a, \tilde{a}). \quad (\text{C.9})$$

The explicit expressions are given by

$$\Gamma_{\text{WZW}}(a - \tilde{a}) = -\frac{1}{8\pi} \int dx^+ dx^- [\partial_+ (u - \tilde{u}) \partial_- (u - \tilde{u}) - \partial_+ (u - \tilde{u}) \partial_- (v - \tilde{v})], \quad (\text{C.10})$$

where:

$$a_+ = \partial_+ u, \quad a_- = \partial_+ v, \quad \tilde{a}_+ = \partial_+ \tilde{u}, \quad \tilde{a}_- = \partial_+ \tilde{v}. \quad (\text{C.11})$$

The Chern-Simons action  $S_{\text{CS}}(\tilde{a})$  can be simplified to

$$S_{\text{CS}}(\tilde{a}) = -\frac{1}{8\pi} NM \int dx^+ dx^- d\rho \left( \tilde{a}_+ \tilde{f}_{-\rho} + \tilde{a}_- \tilde{f}_{\rho+} + \tilde{a}_\rho \tilde{f}_{+-} \right), \quad (\text{C.12})$$

where  $\rho$  is the radial direction away from the  $N$  D3-branes and we have restricted the RR flux to  $s$ -waves on the  $S^5$ . Likewise

$$S_{\text{CS}}(a, \tilde{a}) = \frac{1}{8\pi} NM \int dx^+ dx^- \left( \tilde{f}_{+-}(0) \int dr a_r + f_{+-}(0) \int d\rho \tilde{a}_\rho \right), \quad (\text{C.13})$$

where  $f_{+-}(0)$  and  $\tilde{f}_{+-}(0)$  are the boundary values of  $f_{+-}$  and  $\tilde{f}_{+-}$  respectively and  $r$  is the radial coordinate away from the  $D7$ -branes. Note that the path integral is Gaussian

and it is enough to evaluate the action on the equations of motion. Since we have both bulk and boundary contributions to the action we need to solve the equations of motion separately on the bulk and on the boundary.

The the bulk equations of motion yield:

$$\tilde{f}_{-\rho} = 0, \quad \tilde{f}_{+\rho} = 0, \quad 2\tilde{f}_{+-} = \tilde{f}_{+-}(0). \quad (\text{C.14})$$

Furthermore, the boundary equations of motion give:

$$\int dr a_r = -u, \quad 2\tilde{f}_{+-}(0) = f_{+-}(0). \quad (\text{C.15})$$

Evaluating the action on this solution gives:

$$\Gamma(a, \tilde{a})|_{\text{solution}} = \Gamma_{\text{WZW}}(a). \quad (\text{C.16})$$

Therefore, the final result of performing the path integral (C.1) is:

$$Z = \exp [i(M\Gamma_{\text{WZW}}(A) + MN\Gamma_{\text{WZW}}(a))]. \quad (\text{C.17})$$

We can now combine the  $SU(N)$  connection  $A$  with the  $U(1)$  connection  $a$  into a  $U(N)$  gauge field, which with some abuse of notation we will also denote by  $A$ .

Therefore, integrating out the localized fields together with the non-dynamical gauge fields on the  $D7$ -branes has the effect of inserting the following surface operator into the  $\mathcal{N} = 4$  SYM path integral in the  $D7$ -brane background:

$$Z = \exp (iM\Gamma_{\text{WZW}}(A)). \quad (\text{C.18})$$

#### D. A probe $D7$ -brane in $AdS_5 \times S^5$

In this appendix we study the supersymmetries preserved by the  $D7$ -brane in  $AdS_5 \times S^5$  which represents a surface operator in the probe approximation.

We consider the following parametrization for  $AdS_5 \times S_5$  (we fix the radius  $L = 1$ )

$$ds_{AdS \times S}^2 = \rho^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{d\rho^2}{\rho^2} + d\theta^2 + \sin^2 \theta d\Omega_4^2, \quad (\text{D.1})$$

where the metric on  $S^4$  is given by:

$$d\Omega_4^2 = d\varphi_1^2 + \sin^2 \varphi_1^2 d\varphi_2^2 + \sin^2 \varphi_1^2 \sin^2 \varphi_2^2 d\varphi_3^2 + \sin^2 \varphi_1^2 \sin^2 \varphi_2^2 \sin^2 \varphi_3^2 d\varphi_4^2. \quad (\text{D.2})$$

It is useful to introduce tangent space gamma matrices, i.e.  $\gamma_{\underline{m}} = e_{\underline{m}}^m \Gamma_m$  ( $m, \underline{m} = 0, \dots, 9$ ) where  $e_{\underline{m}}^m$  is the inverse vielbein and  $\Gamma_m$  are the target space matrices:

$$\begin{aligned} \gamma_\mu &= \frac{1}{\rho} \Gamma_\mu \quad (\mu = 0, 1, 2, 3), & \gamma_4 &= \rho \Gamma_\rho, & \gamma_5 &= \Gamma_\theta, \\ \gamma_{a+5} &= \frac{1}{\sin \theta} \left( \prod_{j=1}^{a-1} \frac{1}{\sin \varphi_j} \right) \Gamma_{\varphi_a} \quad (a = 1, 2, 3, 4) \end{aligned} \quad (\text{D.3})$$

The Killing spinor of  $AdS_5 \times S^5$  in the coordinates (D.1) is given by [41]

$$\epsilon = \left[ -\rho^{-\frac{1}{2}} \gamma_4 h(\theta, \varphi_a) + \rho^{\frac{1}{2}} h(\theta, \varphi_a) (\eta_{\mu\nu} x^\mu \gamma^\nu) \right] \eta_2 + \rho^{\frac{1}{2}} h(\theta, \varphi_a) \eta_1 \quad (D.4)$$

where:

$$h(\theta, \varphi_a) = e^{\frac{1}{2}\theta\gamma_{45}} e^{\frac{1}{2}\varphi_1\gamma_{56}} e^{\frac{1}{2}\varphi_2\gamma_{67}} e^{\frac{1}{2}\varphi_3\gamma_{78}} e^{\frac{1}{2}\varphi_4\gamma_{89}}. \quad (D.5)$$

$\eta_1$  and  $\eta_2$  are constant ten dimensional complex spinors satisfying

$$\gamma_{11}\eta_1 = -\eta_1 \quad \gamma_{11}\eta_2 = \eta_2 \quad (D.6)$$

with  $\gamma_{11} = \gamma_0\gamma_1 \dots \gamma_9$ . They also satisfy

$$\tilde{\gamma}\eta_1 = \eta_1 \quad \tilde{\gamma}\eta_2 = -\eta_2, \quad (D.7)$$

where  $\tilde{\gamma} = i\gamma^{0123}$  is the four dimensional chirality matrix. Thus, each spinor  $\eta_{1,2}$  has 16 independent real components.

The supersymmetries preserved by the embedding of a  $D$ -brane probe, are those that satisfy

$$\Gamma_\kappa \epsilon = \epsilon, \quad (D.8)$$

where  $\Gamma_\kappa$  is the  $\kappa$ -symmetry transformation matrix of the probe worldvolume theory and  $\epsilon$  is the Killing spinor of the  $AdS_5 \times S^5$  background (D.4). Both  $\Gamma_\kappa$  and  $\epsilon$  have to be evaluated at the location of the probe.

Let's now consider a  $D7$ -brane embedding with an  $AdS_3 \times S^5$  worldvolume geometry, with embedding:

$$\begin{aligned} \sigma^0 = x^0 & & \sigma^1 = x^1 & & \sigma^2 = \rho & & \sigma^3 = \theta & & \sigma^{3+a} = \varphi^a & (a = 1, 2, 3, 4) \\ & & x^2 = 0 & & x^3 = 0 & & & & & (D.9) \end{aligned}$$

and with the worldvolume gauge field set to zero. The matrix  $\Gamma_\kappa$  for a  $D7$ -brane in a background with zero  $B$ -field and dilaton is given by

$$d^8\sigma \Gamma_{D7} = \frac{1}{\sqrt{-\det(g_{ij})}} \Gamma_{(8)} I \quad (D.10)$$

where  $\Gamma_{(8)} = \frac{1}{8!} \Gamma_{i_1 \dots i_8} d\sigma^{i_1} \wedge \dots \wedge d\sigma^{i_8}$  and  $I$  acts on a spinor  $\psi$  by  $I\psi = -i\psi$ . Considering the embedding in (D), the matrix in (D.10) reduces to:

$$\Gamma_{D7} = \gamma_{01456789} I. \quad (D.11)$$

The equation (D.8) has to be satisfied at every point on the worldvolume. Thus, the term proportional to  $\rho^{\frac{1}{2}}$  gives:

$$\Gamma_{D7} h(\theta, \varphi_a) \eta_1 = h(\theta, \varphi_a) \eta_1. \quad (D.12)$$

The terms proportional to  $\rho^{-\frac{1}{2}}$ ,  $\rho^{\frac{1}{2}} x_0$  and  $\rho^{\frac{1}{2}} x_1$  give:

$$\Gamma_{D7} h(\theta, \varphi_a) \eta_2 = -h(\theta, \varphi_a) \eta_2. \quad (D.13)$$

Using

$$h^{-1}\gamma_{014}h = n^I\gamma_{01I} \quad h^{-1}\gamma_{56789}h = n^I\gamma_{I456789} \quad I = 4, 5, 6, 7, 8, 9 \quad (\text{D.14})$$

where

$$n^I(\theta, \varphi_1, \varphi_2, \varphi_3, \varphi_4) = \begin{pmatrix} \cos \theta \\ \sin \theta \cos \varphi_1 \\ \sin \theta \sin \varphi_1 \cos \varphi_2 \\ \sin \theta \sin \varphi_1 \sin \varphi_2 \cos \varphi_3 \\ \sin \theta \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \cos \varphi_4 \\ \sin \theta \sin \varphi_1 \sin \varphi_2 \sin \varphi_3 \sin \varphi_4 \end{pmatrix} \quad (\text{D.15})$$

is a unit vector in  $\mathbb{R}^6$  (that is  $n^I n^I = 1$ ) we get:

$$\begin{aligned} h^{-1}\Gamma_{D7}h &= n^I n^J \gamma_{01I} \gamma_{J456789} I \\ &= -i\gamma_{01456789} \\ &= \gamma^{01} \tilde{\gamma} \gamma_{11}. \end{aligned} \quad (\text{D.16})$$

Thus, the equations (D.12), (D.13) reduce to

$$\gamma^{01}\eta_1 = -\eta_1 \quad \gamma^{01}\eta_2 = \eta_2 \quad (\text{D.17})$$

Since  $\eta_1$  and  $\eta_2$  satisfy (D.6) and (D.7), they can be written in terms of ten dimensional Majorana-Weyl spinors  $\epsilon$  and  $\varepsilon$  of negative and positive chirality respectively:

$$\eta_1 = \epsilon + i\gamma^{0123}\epsilon \quad \eta_2 = \varepsilon - i\gamma^{0123}\varepsilon. \quad (\text{D.18})$$

By evaluating the Killing spinor (D.4) near the boundary,  $\epsilon$  can be identified with the generator of Poincare supersymmetry while  $\varepsilon$  can be identified with the generator of conformal supersymmetry of  $\mathcal{N} = 4$  SYM. Thus the equations (D.17) become:

$$\gamma^{01}\epsilon = -\epsilon, \quad \gamma^{01}\varepsilon = \varepsilon. \quad (\text{D.19})$$

These conditions are equivalent to (3.6) and (3.7), which describe the unbroken Poincare and conformal supersymmetries respectively in the field theory. Therefore, for coincident  $D7$  branes we have shown that they preserve the same half of the Poincare and conformal supersymmetries as the field theory does in the probe approximation.

### D.1 $D7$ probe without conformal supersymmetries

The  $D7$ -brane embedding we have just discussed solution can be generalized to the case when  $x^2 = \bar{x}^2$  and  $x^3 = \bar{x}^3$  where  $\bar{x}^2$  and  $\bar{x}^3$  are arbitrary constants. The bosonic symmetry of this embedding is  $ISO(1, 1) \times SO(6)$ . We note that the conformal and  $U(1)$  symmetries are broken in the case of separated  $D7$ -branes just like in the field theory.

In this case, the matrix (D.10) is still given by (D.11). The supersymmetry conditions are

$$\Gamma_{D7}h(\theta, \varphi_a)\eta_1 = h(\theta, \varphi_a)\eta_1 \quad (\text{D.20})$$

$$\Gamma_{D7}h(\theta, \varphi_a)\eta_2 = -h(\theta, \varphi_a)\eta_2 \quad (\text{D.21})$$

$$\Gamma_{D7}h(\theta, \varphi_a)\eta_2 = h(\theta, \varphi_a)\eta_2. \quad (\text{D.22})$$

The equations (D.21) and (D.22) imply that the conformal supersymmetries are completely broken. The equation (D.20) implies that the preserved Poincare supersymmetries satisfy:

$$\gamma^{01}\epsilon = -\epsilon. \quad (\text{D.23})$$

When  $\bar{x}^2 = \bar{x}^3 = 0$  the equation (D.22) does not have to be satisfied and half of the conformal supersymmetries are preserved. We thus recover the symmetries preserved by the field theory in the probe approximation.

## E. The Killing spinor

The goal of this appendix is to construct the Killing spinor of the geometry dual to the surface operator. The geometry can be written as follows

$$ds^2 = -H_7^{-1/2}H_3^{-1/2}dx^+dx^- + H_7^{-1/2}H_3^{1/2}d\rho^2 + H_7^{-1/2}d\Omega_5 + H_7^{1/2}H_3^{-1/2}dzd\bar{z} \quad (\text{E.1})$$

$$F_{0123\rho} = H_7\partial_\rho H_3^{-1}, \quad (\text{E.2})$$

where

$$H_3 = \frac{L^4}{\rho^4} \quad (\text{E.3})$$

and  $H_7$  is the harmonic function of the  $D7$ -brane solution. To find the Killing spinor we substitute the above solution into the gravitino and dilatino variations, which in the presence of one- and five-form fluxes take the form:

$$\delta\Psi_M = \partial_M\epsilon + \frac{1}{4}\omega_M^{AB}\Gamma_{AB}\epsilon - \frac{i}{8}e^\Phi\partial_N C\Gamma^N\Gamma_M\epsilon - \frac{i}{8\cdot 5!}e^\Phi F_{M_1\dots M_5}\Gamma^{M_1\dots M_5}\Gamma_M\epsilon = 0, \quad (\text{E.4})$$

$$\delta\psi = (\Gamma^M\partial_M\Phi)\epsilon + ie^\Phi\partial_M C\Gamma^M\epsilon = 0. \quad (\text{E.5})$$

The dilatino variation is independent of the five-form flux and gives

$$\tau = \tau(z), \quad \gamma_{\bar{z}} = 0, \quad (\text{E.6})$$

as in the case of  $D7$ -brane solutions. When we substitute (E.1) into the gravitino variation, there will be terms proportional to  $\partial H_7$  and terms proportional to  $\partial H_3$  which will essentially separate. The term with  $\partial H_7$  cancel if  $\epsilon \sim H_7^{-1/8}$  and (E.6) is satisfied exactly like in the case of  $D7$ -brane solutions. Let us concentrate on the terms proportional to  $\partial H_3$ . Let us first consider the variation  $\delta\Psi_z$ . We obtain:

$$\frac{H_7^{1/2}}{8H_3^{3/2}}\partial_\rho H_3\gamma_4\gamma_z(\epsilon + i\gamma_0\gamma_1\gamma_2\gamma_3\epsilon) = 0. \quad (\text{E.7})$$

Note that  $\partial_z\epsilon$  cancels against the terms proportional to  $\partial_z H_7$  and, hence, eq. (E.7) is not a differential equation on  $\epsilon$ . To satisfy (E.7) we have to require:

$$i\gamma^{0123}\epsilon = \epsilon. \quad (\text{E.8})$$

Eqs. (E.6) and (E.8) are equivalent to (2.5) and (2.6) and, hence,  $\epsilon$  has eight independent components corresponding to eight preserved supercharges. This is in agreement



with our field theory discussions. Now we consider the equation  $\delta\Psi_{\pm} = 0$ . Due to the restriction (E.8), it follows that

$$\delta\Psi_{\pm} = \partial_{\pm}\epsilon = 0. \tag{E.9}$$

This means that  $\epsilon$  is independent of  $x^{\pm}$ . Similarly, from the equation  $\delta\Psi_{\rho} = 0$  we obtain

$$\partial_{\rho}\epsilon - \frac{1}{2\rho}\epsilon = 0, \tag{E.10}$$

which implies  $\epsilon \sim \rho^{1/2}$ . The last equations to consider is  $\delta\Psi_a = 0$ , where  $\Psi_a$  are the components of the gravitino along  $S^5$ . These equations are

$$D_a\epsilon - \frac{1}{2}\gamma_4\Gamma_a\epsilon = 0. \tag{E.11}$$

These are the standard equations for the Killing spinor on  $S^5$  [9, 10]. The solution is given in terms of the operator  $h(\theta, \varphi_a)$  defined in (D.5). Combining the above conclusions we find that the Killing spinor is given by

$$\epsilon = h(\theta, \varphi_a)H_7^{-1/2}\rho^{1/2}\epsilon_0, \tag{E.12}$$

where both  $\epsilon$  and  $\epsilon_0$  satisfy conditions (2.5) and (2.6) (note that  $\gamma_+$  and  $\gamma_{\bar{z}}$  commute with  $h(\theta, \varphi_a)$ ).

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